spprob8.tex

Problems 8. 3.12.2010

Q1. (i) If $M = (M_t)$ is a mg, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submg.
(ii) If in (i) M is a submg, and ϕ is also non-decreasing on the range of M, show that again $\phi(M)$ is a submg.

Q2. (i) Deduce from Q1 that for $B = (B_t)$ BM, $B^2 = (B_t^2)$ is a submg. Find the increasing process in its Doob-Meyer decomposition. Deduce that BM has quadratic variation t.
(ii) Show that for $p \geq 1$ $|B|^p$ is a submg.
(iii) Show that B^+ is a submg.

Q3. *Martingale transforms (Burkholder).* If $X = (X_n)$ is a mg [submg, supermg], $C = (C_n)$ is predictable, write

$$(C \bullet X)_n := \sum_1^n C_k(X_k - X_{k-1})$$

($C \bullet N$ is the *martingale* [submg, supermg] *transform* of X by C). Show that
(i) if C is bounded and non-negative and X is a submg [supermg], $C \bullet X$ is a submg [supermg] null at 0;
(ii) if C is bounded and X is a mg, $C \bullet X$ is a mg null at 0.

Upcrossings. For a process X and interval $[a, b]$, define stopping times σ_k, τ_k by $\sigma_1 := \min\{n : X_n \leq a\}$, $\tau_1 := \min\{n > \sigma_1 : X_n \geq b\}$, and inductively $\sigma_k := \min\{n > \tau_{k-1} : X_n \leq a\}$, $\tau_k := \min\{n > \sigma_k : X_n \geq b\}$. Call $[\sigma_k, \tau_k]$ an *upcrossing* of $[a, b]$ by X, and write $U_n := U_n([a, b], X)$ for the number of such upcrossings by time n.

Q4. *Upcrossing Inequality (Doob).* If X is a submg, show that

$$EU_n([a, b], X) \leq E[(X_n - a)^+]/(b - a).$$

Q5. *Martingale Convergence Theorem (Doob).* Show that an L_1-bounded submg $X = (X_n)$ (i.e. $E|X_n| \leq K$ for some K and all n) is a.s. convergent.

Q6. (i) Show that a non-negative supermg is [a.s.] convergent.
(ii) Give an example of a non-convergent submg [so L_1-boundedness, or some such condition, is needed in Q5].

NHB