spsoln1.tex

**Solutions 1. 22.11.2010**

1. Being either 1 on the set or 0 off it, an indicator is determined by its parity (odd = 1, even = 0), or its value modulo 2.

$$I_{A\Delta B} = I_A + I_B - 2IA \cap B = I_A + I_B = I_{B\Delta A} \qquad (\mathrm{mod}\ 2).$$

For, on $A \setminus B$ and $B \setminus A$ both sides are 1; both sides are 0 on $A^c \cap B^c$ and 0 and 1+1-2 = 0 on $A \cap B$. Since addition (hence also addition mod 2) is associative, $\Delta$ is associative:

$$I_{A\Delta(B\Delta C)} = I_A + I_{B\Delta C} = I_A + I_B + I_C = I_{(A\Delta B)\Delta C} \quad \mathrm{mod}\ 2,$$

so $A\Delta(B\Delta C) = (A\Delta B)\Delta C$. So we may write either side as $A\Delta B\Delta C$ omitting brackets, and similarly for $A_1\Delta \ldots \Delta A_n$.

Assume by induction that $A_1\Delta \ldots \Delta A_n = \{x : x$ is in an odd number of the sets$\}$. Then $A_1\Delta \ldots \Delta A_{n+1} = (A_1\Delta \ldots \Delta A_n)\Delta A_{n+1}$ is the set of points in an even number of the first $n$ sets and the last, or an odd number of the first $n$ and not the last, i.e. is the set of points in an odd number of the first $n+1$ sets, completing the induction.

Since $I_\emptyset = 0$, $A\Delta\emptyset = \emptyset\Delta A = A$, for all $A$. Combining: the set $\mathcal{P}(\Omega)$ of all subsets of $\Omega$ is an additive abelian group under $\Delta$, with $\emptyset$ as 0 element.

Since $I_A.I_B = I_{A\cap B}$, $\mathcal{P}(\Omega)$ is an associative system with $\cap$ as multiplication. Since $A \cap \Omega = A = \Omega \cap A$, $\Omega$ serves as identity, 1. Since

$$I_{A\cap(B\Delta C)} = I_A.I_{B\cap C} = I_A(I_B + I_C) = I_{A\cap B} + I_{A\cap C} = I_{(A\cap B)\Delta(A\cap C)} \quad (\mathrm{mod}\ 2),$$

$$A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C),$$

showing that $\cap$ as multiplication is distributive over $\Delta$ as addition. Combining, $\mathcal{P}(\Omega)$ is a ring under these operations (called the *Boolean ring*), with $\emptyset$ as 0 and $\Omega$ as 1. Since $A\Delta A = \emptyset$, each set $A$ is its own additive inverse.

*Note*. 1. A ring in Algebra is a set with two operations, called addition and multiplication, which is (a) an abelian group under addition, (b) an associative system (not necessarily commutative) under multiplication, and the distributive law holds. Prototypes: integers; polynomials; [square] matrices (non-commutative).

2. *Boolean algebra* (below) is important in Mathematical Logic and Computer Science [membership of a set corresponds to truth of a proposition; recall the use of Boolean variables in do-loops in computer programming]. A *Boolean algebra* is a ring as above, with multiplication idempotent: $a^2 = a$ for every $a$ (this automatically holds in the above context, as $A \cap A = A$ for every $A$). *Stone's Representation Theorem* (Marshall H. STONE (1903-89) in 1936) states that every Boolean algebra arises as a ring of sets as above (this can be used as a weaker form of the Axiom of Choice).

Q2. (i) When $\limsup x_n$ is finite, $c$ say, it is characterised by the following property: for each $\epsilon > 0$,
(a) $x_n \leq c + \epsilon$ for all large enough $n$; (b) $x_n \geq c - \epsilon$ for infinitely many $n$
(see e.g. W. Rudin, *Principles of mathematical analysis*, 3.17). Applied to indicator functions, which take values 0 and 1 only, the first is no restriction, so can be ignored; the second is
(b') $c(\omega) = 1$ for infinitely many $n$.
(ii) $I_{\limsup A_n}$ is 1 iff infinitely many of the $A_n$ occur, and so by (i) is $\limsup I_{A_n}$, proving the first part.
$I_{\liminf A_n}$ is 1 iff all the $A_n$ occur from some point on, and so is $\liminf I_{A_n}$, proving the second part.
*Note*. 1. In Real Analysis, lim sup and lim inf, together with $O$ and $o$, are powerful tools enabling one to strip proofs of superfluous $\epsilon$s. One should never use an $\epsilon$ except when it is genuinely needed – which is usually in the hard proofs.
(ii) We will meet $\limsup A_n$ later in connection with the Borel-Cantelli lemmas [II.8, L13].

Q3. If the sets are $\{A_n\}_{n=1}^{\infty}$ and $A_n = \{x_{n,k}\}_{k=1}^{\infty}$, display the pooint $x_{n,k}$ at the point $(n, k)$ in the first quadrant. By 'diagonal sweep', enumerate this double sequence in a single sequence $(x_{1,1}; x_{2,1}, x_{1,2}; x_{3,1}, x_{2,2}, x_{1,3}; \ldots)$. This shows that $\cup A_n$ is countable [this proof is due to Cantor, who used it to show that the rationals are countable].

Q4. If $A_n$ are $\mu$-null,

$$\mu(\cup_1^{\infty} A_n) \leq \sum_1^{\infty} \mu(A_n) = \sum_1^{\infty} 0 = 0.$$

NHB