spsoln10.tex

Solutions 10. 17.12.2010

Q1. (i)

$$\begin{aligned}
\psi(t) = E[e^{itY}] &= E[\exp\{it(X_1 + \ldots + X_N)\}] \\
&= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n].P(N = n) \\
&= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1 + \ldots + X_n)\}] \\
&= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n \\
&= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\
&= \exp\{-\lambda(1 - \phi(t))\}.
\end{aligned}$$

Differentiate:

$$\psi'(t) = \psi(t).\lambda\phi'(t),$$

$$\psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives $EY = \lambda\mu$. Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2].$$

(ii) Given N, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As N is Poisson with parameter λ, N has mean λ and variance λ. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2].$$

Q2 . (i). Write $f(B,t) := (B^2 - t)^2$. By Itô's formula,

$$df = f_B dB + f_t dt + \frac{1}{2}[f_{BB}(dB)^2 + 2f_{Bt}dBdt + f_{tt}(dt)^2].$$

In the [...] on RHS, $(dB)^2 = dt$, $dBdt = 0$, $(dt)^2 = 0$. Also

$f_B = 2.2B(B^2-t), \qquad f_t = -2(B^2-t), \qquad f_{BB} = 4(B^2-t)+4B.2B = 12B^2-4t.$

So

$df = 4B(B^2-t)dB - 2(B^2-t)dt + (6B^2-2t)dt = 4B(B^2-t)dB + 4B^2dt.$

As $M = f - 4\int_0^t B_s^2 ds$,

$$dM = df - 4B_t^2 dt = 4B(B^2 - t)dB :$$

$$M_t = 4\int_0^t B_s(B_s^2 - s)dB_s.$$

The Itô integral on the RHS is a continuous local martingale starting from 0. Now $B_t =_d t^{1/2}.Z$ where Z is $N(0,1)$. As Z has all moments finite, each $E[B_t^n]$ is a polynomial in t. So the integrand $h = h(B_t,t)$ on RHS satisfies the integrability condition $\int_0^t E[h_s^2]ds < \infty$ for all t. So the RHS is a (true) continuous mg starting from 0.

(ii). With $[M] = ([M_t])$ the quadratic variation of M,

$$d[M]_t = (dM)_t^2; \qquad dM_t = 4B_t(B_t^2 - t)dB_t.$$

So

$$d[M]_t = 16B_t^2(B_t^2 - t)^2(dB_t)^2 = 16B_t^2(B_t^2-t)^2dt :$$

$$[M]_t = 16\int_0^t B_s^2(B_s^2 - s)^2 ds.$$

NHB