spsoln3.tex

Solutions 3. 5.11.2010

Q1. (i) Draw your own picture for 'chord below arc'.
(ii) $-\log\ x$ has second derivative $1/x^2 > 0$ on $(0, \infty)$, so $-\log$ is convex. So

$$\lambda_1 \log\ x_1 + \lambda_2 \log\ x_2 \leq \log(\lambda_1 x_1 + \lambda_2 x_2).$$

Exponentiating,

$$x_1^{\lambda_1} x_2^{\lambda_2} \leq \lambda_1 x_1 + \lambda_2 x_2.$$

(iii) As e^x has second derivative $e^x > 0$, exp is convex. So

$$\exp\{\lambda_1 x_1\}. \ldots . \exp\{\lambda_1 x_1\} = \exp\{\lambda_1 x_1 + \ldots + \lambda_n x_n\} \leq \lambda_1 e^{x_1} + \ldots + \lambda_n e^{x_n}.$$

Take $x_i = \log a_i$:

$$a_1^{\lambda_1}. \ldots . a_n^{\lambda_n} \leq \lambda_1 a_1 + \ldots + \lambda_n a_n.$$

Take each $\lambda_i = 1/n$:

$$(a_1 . \ldots . a_n)^{1/n} \leq (a_1 + \ldots + a_n)/n : \qquad G \leq A. \quad //$$

Q2. Where $|g| \leq |f|^{p-1}$, $|fg| \leq |f|^p$, integrable. Elsewhere, $|g| > |f|^{p-1}$, so $|f| < |g|^{1/(p-1)} = |g|^{q-1}$, so $|fg| < |g|^q$, integrable. Combining, fg is integrable: $fg \in L_1$.

The set where $fg = 0$ makes no contribution, so we can assume fg non-zero, so f, g non-zero, so $\int |f|^p$, $\int |g|^q$ positive. Apply Q1(ii) with $\lambda_1 = 1/p$, $\lambda_2 = 1/q$ (these sum to 1), $x_1 = |f|^p / \int |f|^p$, $x_2 = |g|^q / \int |g|^q$. This gives

$$\frac{|fg|}{(\int |f|^p)^{1/p} (\int |g|^q)^{1/q}} \leq \frac{|f|^p}{p \int |f|^p} + \frac{|g|^q}{q \int |g|^q}.$$

Integrate: the RHS integrates to

$$\frac{1}{p} + \frac{1}{q} = 1.$$

The LHS integrates to

$$\frac{\int |fg|}{(\int |f|^p)^{1/p} (\int |g|^q)^{1/q}}.$$

Combining gives Hölder's inequality. Taking $p = q = 2$ gives the Cauchy-Schwarz inequality.

Q3. If $p = 1$, $|f + g| \le |f| + |g|$ by the Triangle Inequality. Integrating this gives the case $p = 1$ of Minkowski's inequality. So take $p > 1$ below.

If A is the set where $|f| \ge |g|$, so A^c is the set where $|f| < |g|$,

$$|f + g|^p \le 2^p |f|^p \quad \text{on } A, \quad 2^p |g|^p \quad \text{on } A^c.$$

So if $f, g \in L_p$, $f + g \in L_p$, giving (i). For (ii),

$$\int |f+g|^p = \int |f+g|.|f+g|^{p-1} \le \int |f|.|f+g|^{p-1} + \int |g|.|f+g|^{p-1},$$

by the Triangle Inequality. We estimate the first term on the right by Hölder's inequality:

$$\int |f|.|f+g|^{p-1} \le (\int |f|^p)^{1/p}.(\int |f+g|^{(p-1)q})^{1/q} = (\int |f|^p)^{1/p}.(\int |f+g|^p)^{1/q},$$

as $(p-1)q = p$. Similarly for the other term. Combining,

$$\int |f+g|^p \le [(\int |f|^p)^{1/p} + (\int |g|^p)^{1/p}].(\int |f+g|^p)^{1/q}.$$

If $\int |f + g|^p = 0$, there is nothing to prove. If not, divide both sides by $(\int |f+g|^p)^{1/q}$: $1 - 1/q = 1/p$ as p, q are conjugate indices, so

$$(\int |f+g|^p)^{1/p} \le [(\int |f|^p)^{1/p} + (\int |g|^p)^{1/p}],$$

which is Minkowski's inequality. //

Note. 1. These named inequalities are standard, and there are proofs in all the books. See e.g. [S], Th. 12.2 (Hölder), Cor. 12.4 (Minkowski). One needs both a little basic Real Analysis (Schilling uses Young's Inequality, his Lemma 12.1, in place of our use of Jensen's Inequality), and standard properties of the integral.

2. We have deliberately not mentioned the measure μ here, partly to simplify the notation for the proofs (quite fiddly enough as it is), partly to emphasize the full generality. For example, specializing to μ as counting measure we obtain the Hölder, Cauchy-Schwarz and Minkowski inequalities for sums – there is no need to give a separate proof!

NHB