spsoln5.tex

Solutions 5. 19.11.2010

Q1. Take $f(z) := e^{-z^2/2}$. This is entire (has no singularities). So for any contour γ, $\int_\gamma f = 0$, by Cauchy's Residue Theorem (or, use Cauchy's Theorem). Take γ the rectangle with vertices R, $R+iy$, $-R+iy$, $-R$, with sides γ_1 the interval $[-R, R]$, γ_2 the vertical line from R to $R+iy$, γ_3 the horizontal line from $R+iy$ to $-R+iy$, γ_4 the vertical line from $-R+iy$ to $-R$. So $\sum_1^4 \int_{\gamma_i} f = 0$. On γ_2, γ_4: $z = \pm R + iuy$ $(0 \leq u \leq 1)$,

$$f(z) = \exp\{-(\pm R + iuy)^2/2\} = e^{-R^2/2} e^{u^2y^2/2} e^{\pm iRuy} \to 0 \qquad (R \to \infty),$$

as $|e^{\pm iRuy}| = 1$. So $\int_{\gamma_2} f \to 0$, $\int_{\gamma_4} f \to 0$ $(R \to \infty)$. Also $\int_{\gamma_1} f \to \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ as $R \to \infty$). Combining,

$$\int_{\gamma_3} f \to \int_{\infty}^{-\infty} e^{-x^2/2}.e^{y^2/2}.e^{-ixy} dx = -\sqrt{2\pi} \qquad (R \to \infty).$$

So (dividing by $\sqrt{2\pi}$ and by $e^{y^2/2}$, and reversing the direction of integration)

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{-ixy} dx = e^{-y^2/2}.$$

The RHS is real, so the LHS is real. Take complex conjugates:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}.e^{ixy} dx = e^{-y^2/2}.$$

This gives the characteristic function (CF) of the standard normal density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ (the CF is the *Fourier transform* of a probability density).

Q2. (i) If $F(t) := \int_0^\infty e^{-x} \cos xt dx$,

$$\begin{aligned}
F(t) &= \int_0^\infty e^{-x} \cos xt dx = -\int_0^\infty \cos xt de^x \\
&= -[\cos xt.e^{-x}]_0^\infty + \int_0^\infty e^{-x}(-t \sin xt) dx \\
&= 1 = t \int_0^\infty \sin xt de^{-x} \\
&= 1 + t[\sin xt.e^{-x}]_0^\infty - t \int_0^\infty e^{-x}.t \cos xt dx \\
&= 1 - t^2 \int_0^\infty e^{-x} \cos xt dx = 1 - t^2 F(t) :
\end{aligned}$$

$$F(t)(1+t^2) = 1, \qquad F(t) = 1/(1+t^2).$$

Then

$$\int_{-\infty}^{\infty} e^{ixt}.\frac{1}{2}e^{-|x|}dx = \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx + i\int_{-\infty}^{\infty} \sin xt.\frac{1}{2}e^{-|x|}dx$$

$$= \int_{-\infty}^{\infty} \cos xt.\frac{1}{2}e^{-|x|}dx = 1/(1+t^2),$$

by above (the second integral is zero: *odd* integrand, symmetric limits. The first integral is twice $\int_0^\infty$: *even* integrand, symmetric limits.

Thus the characteristic function of the *symmetric exponential* probability density $\frac{1}{2}e^{-|x|}$ is $1/(1+t^2)$.

(ii). Take $\epsilon > 0$. $f(z) = 1/(\pi(1+z^2))$ (to use Jordan's Lemma for $e^{itz}/(\pi(1+z^2)))$. The only singularity inside γ is at $y = i$, a simple pole.

$$Res_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi.2i} = \frac{-ie^{-t}}{2\pi}.$$

By Cauchy's Residue Theorem:

$$\int_\gamma f = 2\pi i.\left(\frac{-ie^{-t}}{2\pi}\right) = e^{-t}.$$

But

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad \text{(Jordan's Lemma)}.$$

This gives the result for $t > 0$. For $t = 0$, it is an arctan (or $\tan^{-1}$) integral. For $t < 0$: replace t by $-t$. //

Thus the CF of the symmetric Cauchy density $1/(\pi(1+x^2))$ is $e^{-|t|}$.

Q3. The similarity between (i) and (ii) of Q2 is an instance of the *Fourier Integral Theorem*: doing the Fourier transform twice gets back to where we started, apart from (a) e^{ixt} first time, but e^{-ixt} the second time; (b) a factor $1/2\pi$. In Q1, the function $e^{-x^2/2}$ is its own Fourier transform (to within the constant factor $1/\sqrt{2\pi}$).

NHB