spsoln5.tex

Solutions 6. 26.11.2010

Q1. Recall our method: one $U(0,1)$ gives a single sequence of independent coin-tosses by dyadic expansion. Rearranging by the Cantor diagonal argument gives infinitely many sequences of coin-tosses , each of which gives a $U(0,1)$, hence a $N(0,1)$, so we get a sequence of $N(0,1)$s, hence a BM.

Repeating the Cantor diagonal argument: each coin-tossing sequence splits into infinitely many. Then proceeding as above, we get infinitely many [independent] BMs.

Q2. As integration is order-preserving, so is conditional expectation. So

$$E[X_n|\mathcal{B}] \leq E[X_{n+1}|\mathcal{B}] \leq E[X|\mathcal{B}].$$

So $E[X_n|\mathcal{B}]$ is increasing, and bounded above, so has a limit; as $E[X_n|\mathcal{B}]$ is $\mathcal{B}$-measurable, so is the limit. We need to show that this limit is $E[X|\mathcal{B}]$. Now for any $B \in \mathcal{B}$,

$$\begin{aligned}
\int_B \lim E[X_n|\mathcal{B}]dP &= \lim \int_B E[X_n|\mathcal{B}]dP \qquad \text{(Monotone Convergence)} \\
&= \lim \int_B X_n dP \qquad \text{(definition of conditional expectation)} \\
&= \int_B X dP \qquad \text{(Monotone Convergence)} \\
&= \int_B E[X|\mathcal{B}]dP \qquad \text{(definition of conditional expectation).}
\end{aligned}$$

As this holds for each $B \in \mathcal{B}$, $\lim E[X_n|\mathcal{B}] = E[X|\mathcal{B}]$ follows.

Q3. Choose any $B \in \mathcal{B}$. By Fatou's Lemma applied to $X_n.I_B$,

$$\int_B \liminf X_n dP = \int \liminf I_B.X_n dP \leq \liminf \int I_B.X_n dP = \liminf \int_B X_n dP.$$

The extreme left and extreme right here and the definition of conditional expectation give

$$\int_B E[\liminf X_n|\mathcal{B}]dP \leq \liminf \int_B E[X_n|\mathcal{B}]dP.$$

As this holds for each $B \in \mathcal{B}$,

$$E[\liminf X_n|\mathcal{B}] \leq \liminf E[X_n|\mathcal{B}].$$

Q4. Choose $B \in \mathcal{B}$. By dominated convergence applied to $X_n I_B$,

$$\int_B X_n dP \to \int_B X dP.$$

By definition of conditional expectation, this says

$$\int_B E[X_n|\mathcal{B}]dP \to \int_B E[X|\mathcal{B}]dP.$$

As this holds for all $B \in \mathcal{B}$,

$$E[X_n|\mathcal{B}] \to E[X|\mathcal{B}].$$

Note. Just as there are conditional versions of the three convergence theorems, there are also conditional versions of the inequalities (Jensen, Hölder, Minkowski). The last two are the most useful, but are most easily proved using *regular conditional probabilities*. Unlike Radon-Nikodym derivatives, which are quite general, these need regularity conditions on the space. These hold in the Euclidean case, which is all we need here. For background and detail, we refer to e.g. [K]:
O. KALLENBERG, *Foundations of modern probability*, 2nd ed., Springer, 2002;
Ch. 5: Conditioning and disintegration.

Q6. The CF of the symmetric Cauchy is $e^{-|t|}$. So $(X_1 + \ldots + X_n)/n$ has CF

$$E \exp\{(X_1 + \ldots + X_n).t/n\} = \Pi_1^n E \exp\{(X_i).t/n\} = [e^{-|t|/n}]^n = e^{-|t|}.$$

So $(X_1 + \ldots + X_n)/n$ is symmetric Cauchy, as required.

This complete failure of $(X_1 + \ldots + X_n)/n$ to converge to the mean μ as $n \to \infty$, as in SLLN, does not contradict the SLLN as here the mean does not exist: the density is $f(x) = 1/(\pi(1 + x^2))$, so $xf(x) \notin L_1$, so the mean does not exist.

NHB