spsoln8.tex

Solutions 8. 10.12.2010

Q1. (i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a mg. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submg.
(ii) If M is a submg, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\phi(M_s) \leq \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s]$$

(the second inequality by conditional Jensen as above), and again $\phi(M)$ is a submg.

Q2. As BM is a mg and x^2 is convex, Q1 (i) gives B^2 a submg. As $B_t^2 - t$ is a mg [L23],

$$B_t^2 = [B_t^2 - t] + t \qquad \text{(submg = mg + incr)}$$

is the Doob-Meyer decomposition of B_t^2, with increasing process t [the QV].
(ii) For $p \geq 1$, $|x|^p$ is convex (for non-zero x, 2nd derivative $p(p-1)|x|^{p-2} \geq 0$).
(iii) $x^+ := \max(x, 0)$ is convex.

Q3. As C is bounded and X is integrable, $C \bullet X$ is integrable; it is null at 0 (empty sum is 0). As C is predictable, C_n is $\mathcal{F}_{n-1}$-measurable, so

$$E[(C \bullet X)_n - (C \bullet X)_{n-1}|\mathcal{F}_{n-1}] = E[C_n(X_n - X_{n-1}|\mathcal{F}_{n-1}] = C_n E[X_n - X_{n-1}|\mathcal{F}_{n-1}],$$

taking out what is known. This is ≥ 0 in case (i) with $C \geq 0$ and X a submg, and 0 in case (ii) with X a mg.

Q4. As $(X - a)^+$ is a submg by Q2 (iii) and upcrossings of $[a, b]$ by X correspond to upcrossings of $[0, b - a]$ by $(X - a)^+$, we may (by passing to $(X - a)^+$) take $X \geq 0$, $a = 0$. Write

$$V_n := \sum_{k \geq 1} I(\sigma_k < n \leq \tau_k).$$

Then V is predictable (this comes from the "$<$" above – we know at time $n-1$ whether the kth upcrossing has begun). So $1-V$ is predictable. So by Q3 the transform $(1-V) \bullet X$ is a submg. So

$$E[(1-V) \bullet X)_n] \geq E[(1-V) \bullet X)_0] = 0 : \qquad E[(V \bullet X)_n] \leq E[X_n].$$

Each completed upcrossing contributes at least b to the sum in $(V \bullet X)n = \sum_1^n V_k(X_k - X_{k-1})$, and the contribution of the last (possibly uncompleted) upcrossing is ≥ 0, so

$$(V \bullet X)_n \geq bU_n.$$

Combining, $bU_n \leq E[(V \bullet X)_n] \leq E[X_n]$: $EU_n \leq E[X_n]/b$. Reverting to the original notation gives the result.

Q5. For $a < b$ rational, the expected number EU_n of upcrossings of $[a, b]$ up to time n is $\leq (K + |a|)/(b-a) < \infty$, for each n. As U_n increases in n, monotone convergence gives $E[\sup U_n] < \infty$. So $U := \sup U_n < \infty$ a.s. If $X_* := \text{liminf} X_n$, $X^* := \text{limsup} X_n$, $\{X_* < X^*\} = \cup_{a,b}\{X_* < a < b < X^*\}$ ($a < b$ rational). Each such set is null (or U would be infinite). So their union is null, i.e. $X_* = X^*$ a.s.: X is a.s. convergent (its limit X_∞ may be infinite). But $E|X| = E[\lim(\inf)|X_n|] \leq \text{liminf} E[|X_n|]$ (by Fatou), $\leq K < \infty$. So $|X_\infty| < \infty$ a.s., and $X_n \to X_\infty$ finite, a.s. //

Q6. (i) If X_n is a supermg, EX_n decreases. As $X \geq 0$, $X_n \geq 0$. So EX_n converges (decreasing and bounded below), so is bounded. So the supermg $-X$ is L_1-bounded, so convergent by Q5, so X is convergent.
(ii) The sum of independent coin-tosses (± 1, prob. 1/2 each) is a mg, but connot converge (so cannot be L_1-bounded).

NHB