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STOCHASTIC PROCESSES: EXAMINATION 2011/12

Answer five questions out of six; 20 marks per question.

Q1. (i) State Fatou's lemma, without proof.
(ii) State and prove the dominated convergence theorem.
(iii) The Gamma function is defined by

$$\Gamma(\alpha) := \int_0^\infty e^{-x} x^{\alpha-1} dx \qquad (\alpha > 0).$$

Show that

$$\int_0^n \Big(1 - \frac{x}{n}\Big)^n x^{\alpha-1} dx \to \Gamma(\alpha) \qquad (n \to \infty).$$

Q2. (i) Show that for pairwise independent random variables with variances, the variance of a sum is the sum of the variances.
(ii) State the first and second Borel-Cantelli lemmas.
(iii) Show that the Second Borel-Cantelli Lemma continues to hold with independence weakened to pairwise independence.
(iv) Why is this important?

Q3. Define infinite divisibility. State without proof the Lévy-Khintchine formula.

Define the symmetric stable law of index $\alpha \in (0, 2]$, and state its characteristic function and Lévy measure without proof.

The symmetric stable law of index $3/2$ is called the *Holtsmark distribution.* Obtain its Lévy measure (you may assume that $\int_0^\infty x^{-3/2} \sin x dx = \sqrt{2\pi}$).

Show that if $X_1, X_2, \ldots$ are independent and have the Holtsmark distribution, then $(\sum_{k=1}^n X_k)/n^{2/3}$ has the same distribution as X_1. Why does this not contradict the Central Limit Theorem?

Q4. Define uniform integrability. For (X_n) uniformly integrable, show that
(i) $E[\liminf X_n] \le \liminf E[X_n] \le \limsup E[X_n] \le E[\limsup X_n]$;
(ii) if $X_n \to X$ a.s., then $X \in L_1$ and $E[X_n] \to E[X]$;
(iii) if $X_n \to X$ in probability, then $X \in L_1$ and $E[X_n] \to E[X]$ (you may assume that in a metric space, convergence of a sequence to ℓ is equivalent

to every subsequence having a further sub-subsequence converging to ℓ).

Q5. Define a *convex function*. State without proof Jensen's inequality, and its conditional form.
(i) If $M = (M_t)$ is a martingale, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submartingale.
(ii) If in (i) M is a submartingale, and ϕ is also non-decreasing on the range of M, show that again $\phi(M)$ is a submartingale.
(iii) Deduce that for $B = (B_t)$ Brownian motion, $B^2 = (B_t^2)$ is a submartingale.
(iv) Find the increasing process in its Doob-Meyer decomposition. Deduce that Brownian motion has quadratic variation t.

Q6. (i) The Ornstein-Uhlenbeck stochastic differential equation is

$$dV = -\beta V dt + \sigma dW. \qquad (OU)$$

Interpret (OU) physically.
(ii) Solve (OU) to obtain

$$V_t = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u.$$

(iii) By using the Itô isometry, or otherwise, show that V_t has distribution $N(v_0 e^{-\beta t}, \sigma^2(1 - e^{-2\beta t})/(2\beta))$.
(iv) By (iii) and independence of Brownian increments, or otherwise, show that the covariance is

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u}(1 - 2e^{-2\beta t})/(2\beta) \qquad (u \geq 0)^1.$$

(v) Show that V is Markov.
(vi) What is the financial relevance of this model?

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[1] There were two unfortunate typos in the question (though not in the solution, or the lecture notes from which this was taken ($e^{-2\beta u}$, and no $/(2\beta)$)), both announced during the exam.