STOCHASTIC PROCESSES: EXAMINATION SOLUTIONS 2011/12

Q1. **Theorem (Fatou's lemma)**. If f_n are measurable and bounded below by an integrable function g, then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Of course, applying the result to $-f_n$ gives the alternative form: for functions bounded above by an integrable function, $\int \limsup \geq \limsup \int$. **[2]**

Theorem (Dominated convergence). If f_n are measurable, $f_n \to f$ and $|f_n| \leq g$ with g μ-integrable, then

$$\int f_n d\mu \to \int f d\mu. \qquad [\mathbf{2}]$$

Proof. First suppose $f_n \geq 0$ and $f_n \to 0$. By Fatou's lemma (both forms),

$$0 = \int 0 d\mu = \int \liminf f_n d\mu \leq \liminf \int f_n d\mu \leq \limsup \int f_n d\mu,$$

$$\limsup \int f_n d\mu \leq \int \limsup f_n d\mu = \int 0.d\mu = 0.$$

Combining these two ($0 = .. \leq ... \leq= 0$), each must be an equality. So $\lim \int f_n d\mu$ exists and is 0.

In the general case, as $|f_n| \leq g$ and $f_n \to f$, $|f| \leq g$; as g is μ-integrable, this gives f μ-integrable (L6, property (viii)). Put $g_n := |f_n - f|$. Then $0 \leq g_n \leq 2g$, $2g$ is μ-integrable, g_n is measurable and tends to 0. So

$$|\int f_n d\mu - \int f d\mu| \leq \int |f_n - f| d\mu \to 0,$$

by the first part applied to $g_n = |f_n - f|$. So $\int f_n d\mu \to \int f d\mu$. // **[10]**

From $1 - x \leq e^{-x}$ $(x > 0)$, $(1 - \frac{x}{n})^n \leq e^{-x}$, and $\to e^{-x}$ as $n \to \infty$. As also $I_{[0,n]}(x) \leq 1$ and $\to 1$, $I_{[0,n]}(x)(1 - \frac{x}{n})^n x^{\alpha-1} \leq e^{-x}x^{\alpha-1}$ and $\to e^{-x}x^{\alpha-1}$. So by dominated convergence,

$$\int_0^n (1 - \frac{x}{n})^n x^{\alpha-1} dx \to \Gamma(\alpha) := \int_0^\infty e^{-x}x^{\alpha-1} dx \qquad (n \to \infty). \qquad [\mathbf{6}]$$

Bookwork seen in lectures; problem unseen.

Q2 (i) *(Variance of sum for Pairwise Independence).*

For X_n pairwise independent, put $S_n := \sum_1^n X_i$, $S := \sum_1^\infty X_i$, $m_n := E[S_n] = \sum_1^n E[X_i]$.

$$var(S_n) = E[(S_n - m_n)^2] = E[(\sum_{i=1}^n (X_i - E[X_i])(\sum_{j=1}^n (X_j - E[X_j])]$$

$$= E[\sum_i \sum_j (\ldots)(\ldots)] = \sum_i E[(\ldots)^2] + \sum_{i \neq j} E(\ldots)(\ldots)] = \sum_i E[(\ldots)^2]$$

(the sum over $i \neq j$ is 0, as there by pairwise independence and the Multiplication Theorem $E[(\ldots)(\ldots)] = E[(\ldots)]E[(\ldots)] = 0.0 = 0$ – variance of sum = sum of variances under pairwise independence). **[5]**

(ii) First Borel-Cantelli Lemma: if $A := limsup A_n$, $\sum P(A_n) < \infty$ implies $P(A) = P(A_n i.o.) = 0$. **[2]**

Second Borel-Cantelli Lemma: if the A_n are independent, $P(A_n) = \infty$ implies $P(A) = P(A_n i.o.) = 1$. **[2]**

(iii) As $I(A_i)$ is Bernoulli with parameter $P(A_i)$, its variance is $P(A_i)[1 - P(A_i)] \leq P(A_i)$. So

$$var(S_n) = E[(S_n - m_n)^2] \leq \sum_1^n P(A_i) = m_n,$$

which increases to $+\infty$ as $\sum P(A_n)$ diverges, by assumption. But

$$\begin{aligned} P(S \leq m_n/2) &\leq P(S_n \leq m_n/2) \qquad (S_n \leq S) \\ &= P(S_n - m_n \leq -m_n/2) \\ &\leq P(|S_n - m_n| \geq m_n/2) \\ &\leq \frac{4}{m_n^2} var(S_n) \qquad \text{(by Tchebycheff's Inequality)} \\ &\leq 4/m_n \quad \text{(by above)} \quad \to 0 \quad (n \to \infty). \end{aligned}$$

But the LHS increases to $P(S < \infty)$, by continuity ($= \sigma$-additivity) of $P(.)$. So $P(S < \infty) = 0$: $P(\sum I(A_n) < \infty) = 0$, i.e. $P(\sum I(A_n) = \infty) = 1$. This says that $P(A_n \ i.o.) = 1$: $P(\limsup A_n) = 1$. // **[8]**

(iv) This is important, as ((i) and (iii) show that the Etemadi proof of the Strong Law of Large Numbers (based on a geometric subsequence – presented in lectures) extends from independence to pairwise independence. **[3]**

All seen in lectures.

Q3. X is *infinitely divisible* (id) if for each n X has the same distribution as the sum of n independent copies of some random variable. **[2]**
The *Lévy-Khintchine formula* (LK) states that X is id iff its characteristic function has the form

$$E \exp\{iuX_1\} = \exp\{-\Psi(u)\} \qquad (u \in \mathbf{R})$$

for some $a \in \mathbf{R}$, $\sigma \geq 0$, $\int \min(1, |x|^2 \mu(dx) < \infty$, where the *Lévy exponent* Ψ is

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI(|x| < 1)\mu(dx). \qquad \mathbf{[2]}$$

X is *symmetric stable* if X has the same distribution as $-X$, and the same distribution as $c(X_1 + X_2)$, where X_1, X_2 are independent copies X_1, X_2 of itself, for some constant c. **[2]**
The Lévy exponent of a symmetric stable law has the form $\Psi(u) = |u|^\alpha$, where $0 < \alpha \leq 2$ is the index. This is Gaussian with $\sigma = 1$ if $\alpha = 2$; if $\alpha \in (0, 2)$ it has Lévy measure $\mu(dx) = cdx/|x|^{1+\alpha}$. **[2]**
So for $\alpha = 3/2$,

$$\Psi(u) = |u|^{3/2} = c(\int_{-\infty}^{-1} + \int_1^\infty)(1-e^{iux})dx/|x|^{5/2} + c(\int_{-1}^0 + \int_0^1)(1-e^{iux}+iux)dx/|x|^{5/2}.$$

In each of $\int_{-\infty}^{-1}$, $\int_{-1}^0$, replace x by $-x$, and use $e^{iux} + e^{-iux} = 2\cos ux$. The $\pm iux$ terms cancel, and then the $\int_0^1$ and $\int_1^\infty$ combine, to give

$$|u|^{3/2} = c.2 \int_0^\infty (1 - \cos ux)dx/x^{5/2}.$$

Take $u > 0$, and differentiate: $\frac{3}{2}u^{1/2} = c.2 \int_0^\infty \sin uxdx/x^{3/2}, = c.2u^{1/2} \int_0^\infty \sin vdv/v^{3/2}$, putting $v := ux$. The integral on the right is $\sqrt{2\pi}$, given. So for $u > 0$

$$\frac{3}{2}u^{1/2} = 2\sqrt{2\pi}cu^{1/2} : \qquad c = \frac{3}{4\sqrt{2\pi}}.$$

The Lévy measure of the Helmholtz distribution is thus

$$\mu(dx) = \frac{3}{4\sqrt{2\pi}}dx/|x|^{5/2}. \qquad \mathbf{[8]}$$

If $X_1, X_2, \ldots$ are independent Helmholtz, the CF of each of $X_1, \ldots, X_n$ is $\exp\{-|u|^{3/2}\}$, so that of each $X_i/n^{2/3}$ is $e^{-|u|^{3/2}/n}$, so that of their sum is $e^{-|u|^{3/2}}$, the Helmholtz CF. So $(X_1 + \ldots + X_n)/n^{2/3}$ is Helmholtz, for each n. This does not contradict the CLT, as the Helmholtz does not have a variance. **[4]**
All seen.

Q4. *Uniform Integrability*[1]. Call X_n *uniformly integrable* (UI) if

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \downarrow 0 \qquad (a \uparrow \infty). \qquad \mathbf{[2]}$$

Theorem. For (X_n) UI,
(i) $E[\liminf\ X_n] \le \liminf E[X_n] \le \limsup E[X_n] \le E[\limsup\ X_n]$.
(ii) If $X_n \to X$ a.s. or in prob., then $X \in L_1$ and $E[X_n] \to E[X]$.

Proof. (i) For $c \ge 0$,

$$E[X_n] = \int X_n dP = \int_{\{X_n < -c\}} X_n dP + \int_{\{X_n \ge -c\}} X_n dP.$$

Choose $\epsilon > 0$. By UI, we can take c so large that each first term on RHS has modulus $< \epsilon$. As $X_n I(X_n \ge -c) \ge -c$, integrable, Fatou's Lemma gives

$$\liminf \int_{\{X_n \ge -c\}} X_n dP \ge \int \liminf X_n I(X_n \ge -c) dP.$$

As $X_n I(X_n \ge -c) \ge X_n$, RHS $\ge \int \liminf X_n dP$. Combining,

$$\liminf E[X_n] \ge E[\liminf X_n] - \epsilon.$$

As $\epsilon > 0$ is arbitrarily small, this gives the 'liminf' part; the 'limsup' part is similar. **[10]**
(ii) If $X_n \to X$ a.s., (ii) follows from (i). **[2]**
If $X_n \to X$ in probability, there is a subsequence $X_{n_k} \to X$ a.s. (quote). Then by (i), $X \in L_1$, and $E[X_{n_k}] \to E[X]$. Similarly, every subsequence has a further sub-subsequence $\to X$ a.s., along which the mean converges to $E[X]$. But this implies convergence along the whole sequence (given). // **[6]**
All seen in lectures.

[1] The solution below is taken from my lecture notes, and is as seen by my checker and the external examiner. Unfortunately, it was not until just before beginning the marking that I realised that the question of the integrability of the limit was not addressed explicitly – as it was similarly passed over in the textbook source I used, Ash Th. 7.5.2. I have given all the marks due for this to any candidate who attempted the question. The best way to do this is to make the relevant integrability an explicit part of the statement of Fatou's Lemma (or Theorem), and I have done this in the relevant lecture, L8, now up on the website. For the textbook I used here, see
V. I. BOGACHEV, *Measure Theory*, Volume 1, Springer 2007, Th. 2.8.3 p.131.

Q5. A function ϕ is *convex* if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \qquad \forall \lambda \in [0,1], x, y. \qquad \textbf{[2]}$$

Jensen's inequality states that

$$\phi(E[X]) \leq E[\phi(X)]$$

for convex ϕ and random variables X with X, $\phi(X)$ both integrable. **[2]**
The *conditional Jensen inequality* states that for $\mathcal{A}$ a σ-field, ϕ, X as above,

$$\phi(E[X|\mathcal{A}]) \leq E[\phi(X)|\mathcal{A}]. \qquad \textbf{[2]}$$

(i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a martingale. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submartingale. **[3]**
(ii) If M is a submartingale, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\begin{aligned} \phi(M_s) &\leq \phi(E[M_t|\mathcal{F}_s]), \\ &\leq E[\phi(M_t)|\mathcal{F}_s] \end{aligned}$$

by the conditional Jensen inequality again, and again $\phi(M)$ is a submartingale. **[4]**
(iii) As Brownian motion B is a martingale (lectures), and x^2 is convex (its second derivative is $1 \geq 0$), B^2 is a submartingale by (i). **[3]**
(iv) As $B_t^2 - t$ is a martingale (which you may quote here as it is not asked – but is easy to prove, as in lectures)

$$B_t^2 = [B_t^2 - t] + t \qquad (\text{submg} = \text{mg} + \text{incr})$$

is the Doob-Meyer decomposition of B_t^2. The increasing process here is t, which is thus the quadratic variation of Brownian motion B. **[4]**
All seen, lectures and problems.

Q6. (i) The Ornstein-Uhlenbeck SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[2]**
(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta C e^{-\beta t} dt + e^{-\beta t} dC = -\beta V dt + e^{-\beta t} dC,$$

so V is a solution of (OU) if $e^{-\beta t} dC = \sigma dW$, $dC = \sigma e^{\beta t} dW$, $C = c + \int_0^t e^{\beta u} dW$. So with initial velocity v_0,

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u. \qquad [\mathbf{4}]$$

(iii) V is Gaussian, as it is obtained from the Gaussian process W by linear operations.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u} dW_u] = \int_0^t e^{\beta u} E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u)^2] = \sigma^2 \int_0^t (e^{-\beta t + \beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t} \int_0^t e^{2\beta u} du = \sigma^2 e^{-2\beta t} [e^{2\beta t} - 1]/(2\beta) = \sigma^2 [1 - e^{-2\beta t}]/(2\beta).$$

So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$. **[4]**
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which (subtracting off $v_0 e^{-\beta t}$ as we may) is

$$\sigma^2 E[e^{-\beta t} \int_0^t e^{\beta v} dW_v . e^{-\beta(t+u)} (\int_0^t + \int_t^{t+u}) e^{\beta w} dW_w].$$

By independence of Brownian increments, the $\int_t^{t+u}$ term contributes 0, leaving as before

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u} [1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty). \quad [\mathbf{4}]$$

(v) The process V is Markov (a diffusion), being the solution of the SDE (OU). **[3]**
(vi) The process shows *mean reversion*, and the financial relevance is to the *Vasicek model* of interest-rate theory. **[3]**
Seen, lectures.

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