spl18.tex

Lecture 18. 18.11.2010

3. Martingale Convergence and Uniform Integrability

Martingale transforms (Burkholder).

If $X = (X_n)$ is a mg [submg, supermg], $C = (C_n)$ is predictable, write

$$(C \bullet X)_n := \sum_1^n C_k(X_k - X_{k-1})$$

($C \bullet N$ is the *martingale* [submg, supermg] *transform* of X by C). Then
(i) if C is bounded and non-negative and X is a submg [supermg], $C \bullet X$ is a submg [supermg] null at 0;
(ii) if C is bounded and X is a mg, $C \bullet X$ is a mg null at 0.
Proof. As C is bounded and X is integrable, $C \bullet X$ is integrable; it is null at 0 (empty sum is 0). As C is predictable, C_n is $\mathcal{F}_{n-1}$-measurable, so

$$E[(C \bullet X)_n - (C \bullet X)_{n-1}|\mathcal{F}_{n-1}] = E[C_n(X_n - X_{n-1}|\mathcal{F}_{n-1}] = C_n E[X_n - X_{n-1}|\mathcal{F}_{n-1}],$$

taking out what is known. This is ≥ 0 in case (i) with $C \geq 0$ and X a submg, and 0 in case (ii) with X a mg. //

Upcrossings.

For a process X and interval $[a, b]$, define stopping times σ_k, τ_k by $\sigma_1 := \min\{n : X_n \leq a\}$, $\tau_1 := \min\{n > \sigma_1 : X_n \geq b\}$, and inductively $\sigma_k := \min\{n > \tau_{k-1} : X_n \leq a\}$, $\tau_k := \min\{n > \sigma_k : X_n \geq b\}$. Call $[\sigma_k, \tau_k]$ an *upcrossing* of $[a, b]$ by X, and write $U_n := U_n([a, b], X)$ for the number of such upcrossings by time n.

Upcrossing Inequality (Doob). If X is a submg,

$$EU_n([a, b], X) \leq E[(X_n - a)^+]/(b - a).$$

Proof. As $(X - a)^+$ is a submg by Q2 (iii) and upcrossings of $[a, b]$ by X correspond to upcrossings of $[0, b - a]$ by $(X - a)^+$, we may (by passing to $(X - a)^+$) take $X \geq 0$, $a = 0$. Write

$$V_n := \sum_{k \geq 1} I(\sigma_k < n \leq \tau_k).$$

Then V is predictable (this comes from the "$<$" above – we know at time $n - 1$ whether the kth upcrossing has begun). So $1 - V$ is predictable. So by above the transform $(1-V) \bullet X$ is a submg. So

$$E[((1-V) \bullet X)_n] \geq E[((1-V) \bullet X)_0] = 0: \qquad E[(V \bullet X)_n] \leq E[X_n].$$

Each completed upcrossing contributes at least b to the sum in $(V \bullet X)_n = \sum_1^n V_k(X_k - X_{k-1})$, and the contribution of the last (possibly uncompleted) upcrossing is ≥ 0, so

$$(V \bullet X)_n \geq bU_n.$$

Combining, $bU_n \leq E[(V \bullet X)_n] \leq E[X_n]$: $EU_n \leq E[X_n]/b$. Reverting to the original notation gives the result. //

(Sub-)Martingale Convergence Theorem (Doob). An L_1-bounded submg $X = (X_n)$ (i.e. $E|X_n| \leq K$ for some K and all n) is a.s. convergent.

Proof. For $a < b$ rational, the expected number EU_n of upcrossings of $[a, b]$ up to time n is $\leq (K + |a|)/(b-a) < \infty$, for each n. As U_n increases in n, monotone convergence gives $E[\sup U_n] < \infty$. So $U := \sup U_n < \infty$ a.s. If $X_* := \liminf X_n$, $X^* := \limsup X_n$, $\{X_* < X^*\} = \cup_{a,b}\{X_* < a < b < X^*\}$ ($a < b$ rational). Each such set is null (or U would be infinite). So their union is null, i.e. $X_* = X^*$ a.s.: X is a.s. convergent (its limit X_∞ may be infinite). But $E|X_\infty| = E[\lim(\inf)|X_n|] \leq \liminf E[|X_n|]$ (by Fatou), $\leq K < \infty$. So $|X_\infty| < \infty$ a.s., and $X_n \to X_\infty$ finite, a.s. //

Corollary (Doob). A non-negative supermg X_n is a.s. convergent.

Proof. As X_n is a supermg, EX_n decreases. As $X \geq 0$, $E[X_n] \geq 0$. So $E[|X_n|] = E[X_n]$ is decreasing and bounded below, so (convergent and) bounded: X_n is L_1-bounded. So the submg $-X_n$ is L_1-bounded, so a.s. convergent by Doob's Theorem, so X_n is a.s. convergent.

Uniform Integrability. Call X_n *uniformly integrable* (UI) if

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \downarrow 0 \qquad (a \uparrow \infty).$$

Note that:
(i) If (X_n) are UI, then each X_n is integrable. For,

$$E|X_n| = \int_{\{|X_n| \leq a\}} |X_n| dP + \int_{\{|X_n|>a\}} |X_n| dP \leq a + o(1) < \infty.$$

(ii) If each $|X_n| \leq Y \in L_1$, then (X_n) is UI.
(iii) If $\sup_n |X_n| \in L_1$, then (X_n) is UI, as then

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \leq \int_{\{|X_n|\geq a\}} (\sup_k |X_k|) dP \to 0 \qquad (a \to \infty),$$

by dominated convergence.

The next result extends Fatou's Lemma and dominated convergence.

Theorem. For (X_n) UI,
(i) $E[\liminf \ X_n] \leq \liminf E[X_n] \leq \limsup E[X_n] \leq E[\limsup \ X_n]$.
(ii) If $X_n \to X$ a.s. or in prob., then $X \in L_1$ and $E[X_n] \to E[X]$.

Proof. (i) For $c \geq 0$,

$$E[X_n] = \int X_n dP = \int_{\{X_n < -c\}} X_n dP + \int_{\{X_n \geq -c\}} X_n dP.$$

Choose $\epsilon > 0$. By UI, we can take c so large that each first term on RHS has modulus $< \epsilon$. As $X_n I(X_n \geq -c) \geq -c$, integrable, Fatou's Lemma gives

$$\liminf \int_{\{X_n \geq -c\}} X_n dP \geq \int \liminf X_n I(X_n \geq -c) dP.$$

As $X_n I(X_n \geq -c) \geq X_n$, RHS $\geq \int \liminf X_n dP$. Combining,

$$\liminf E[X_n] \geq E[\liminf X_n] - \epsilon.$$

As $\epsilon > 0$ is arbitrarily small, this gives the 'liminf' part; the 'limsup' part is similar.
(ii) If $X_n \to X$ a.s., (ii) follows from (i). If $X_n \to X$ in probability, there is a subsequence $X_{n_k} \to X$ a.s. (quote). Then by (i), $X \in L_1$, and $E[X_{n_k}] \to E[X]$. Similarly, every subsequence has a further sub-subsequence $\to X$ a.s., along which the mean converges to $E[X]$. But this implies convergence along the whole sequence (check). //

Uniform integrability is what is needed to pass from a.s. convergence to L_1-convergence, and to strengthen convergence in prob. to a.s. convergence:

Proposition 1. (i) If X_n is UI and a.s. convergent, it is L_1-convergent.
(ii) If $p \in (0, \infty)$, $X_n \to X$ in prob. and $(|X_n|^p)$ is UI, then $X_n \to X$ in L_p.

Proof. (i) For $a > 0$, define $f_a(x)$ as $-a$ for $x \leq -a$, x for $-a \leq x \leq a$, $+a$ for $x \geq a$. Then f_a is bounded and continuous, and (check) $|x - f_a(x)| \leq x$. By the Triangle Inequality,

$$\|X_m - X_n\|_1 \leq \|f_a(X_m) - f_a(X_n)\|_1 + \|X_m - f_a(X_m)\|_1 + \|X_n - f_a(X_n)\|_1.$$

If $X_n \to X_\infty$ a.s., then also $f_a(X_n) \to f_a(X_\infty)$ a.s. as f_a is continuous. As $|f_a| \leq a$, dominated convergence then shows that $f_a(X_n) \to f_a(X_\infty)$ in L_1 (so is Cauchy in L_1). Also

$$\|X_m - f_a(X_m)\|_1 \leq \int_{\{|X_m|>a\}} |X_m| dP$$

by definition of f_a. Let $m, n \to \infty$: the first term on the RHS $\to 0$ as $f_a(X_n)$ is Cauchy in L_1. By UI, the second and third terms $\to 0$ as $a \to \infty$. This shows that X_n is Cauchy in L_1, so convergent in L_1 as L_1 is complete (Riesz-Fischer theorem – quote). //
(ii) We quote this, as we shall not need it; see e.g. Ash [A], Th. 7.5.4.

Proposition 2. (X_n) is UI iff $E[|X_n|]$ is bounded and (X_n) is uniformly absolutely continuous, i.e.

$$\sup_n \int_A |X_n| dP \to 0 \qquad (P(A) \to 0).$$

Proof. If (X_n) is UI,

$$\int_A |X_n| dP = \int_{A\cap\{|X_n|\geq c\}} |X_n| dP + \int_{A\cap\{|X_n|<c\}} |X_n| dP \leq \int_{\{|X_n|\geq c\}} |X_n| dP + cP(A).$$

Choose $\epsilon > 0$. For c large enough, the first term $< \epsilon/2$ for all n. Then if $P(A) < \epsilon/(2c)$, $\int_A |X_n| dP < \epsilon$, proving (X_n) unif. abs. continuous. Also

$$E|X_n| = \int_{\{|X_n|\geq c\}} |X_n| dP + \int_{\{|X_n|<c\}} |X_n| dP < \epsilon + c$$

for large n (the first term by UI), so $E|X_n|$ is bounded.

Conversely, by Markov's Inequality

$$P(|X_n| \geq c) \leq c^{-1} E|X_n| \leq c^{-1} \sup_n E|X_n| \to 0 \qquad (c \to \infty),$$

uniformly in n. This and the uniform absolute continuity give

$$\int_{\{|X_n|\geq c\}} |X_n| dP \to 0 \qquad (c \to \infty)$$

uniformly in n, giving (X_n) UI. //

Lemma (UI Lemma). If $X \in L_1$, then the family $\{E[X|\mathcal{B}]\}$ as $\mathcal{B}$ varies over all sub-σ-fields of $\mathcal{A}$ is UI.

Proof. $|E[X|\mathcal{B}]| \leq E[|X|\ |\mathcal{B}]$. Also, for $a > 0$ $\{|E[X|\mathcal{B}]\ | > a\} \subset \{E[|X|\ |\mathcal{B}] > a\}$, so $I(\{|E[X|\mathcal{B}]\ | > a\}) \leq I(\{E[|X|\ |\mathcal{B}] > a\})$. Multiply:

$$|E[X|\mathcal{B}]\ |I(\{|E[X|\mathcal{B}]\ | > a\}) \leq E[|X|\ |\mathcal{B}]I(\{E[|X|\ |\mathcal{B}] > a\}).$$

Take expectations. Writing $A := \{E[|X|\ |\mathcal{B}] \geq a\}$, the RHS gives $\int_A E[|X|\ |\mathcal{B}]dP$, and as A is $\mathcal{B}$-measurable, this is $\int_A E[|X|]dP$, by definition of conditional expectation. Splitting between $\{|X| \leq b\}$ and $\{|X| > b\}$, this is at most

$$P(E[|X|\ |\mathcal{B}] \geq a) + \int_{\{|X|>b\}} |X|dP.$$

But

$$P(E[|X|\ |\mathcal{B}] \geq a) \leq a^{-1}E[E[|X||\mathcal{B}]]$$

by Markov's Inequality, which is $a^{-1}E|X|$ by the Conditional Mean Formula. Combining,

$$\sup_{\mathcal{B}} \int_A E[|X|\ |\mathcal{B}]dP \leq \frac{b}{a}E|X| + \int_{\{|X|>b\}} |X|dP.$$

Take $b := \sqrt{a}$ and let $a \to \infty$: RHS $\to 0$ (as $X \in L_1$), so LHS $\to 0$. This says that $\{E[X|\mathcal{B}]\}$ is UI, as required. //

Theorem (Lévy). If $Y \in L_1$ and $(\mathcal{F}_n)$ is a filtration with $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then

$$E[Y|\mathcal{F}_n] \to E[Y|\mathcal{F}_\infty] \quad \text{a.s and in } L_1.$$

Proof. If $X_n := E[Y|\mathcal{F}_n]$, then X_n is a mg (w.r.t. $(\mathcal{F}_n)$), and is UI (by the UI Lemma). As $E[|X_n|] \leq E[|Y|] < \infty$, (X_n) is an L_1-bounded mg, so a.s. convergent (Doob's Mg Convergence Thm), to X_∞, say. Also X_n is L_1-convergent, by the Theorem (ii). It remains to show that $X_\infty = E[Y|\mathcal{F}_\infty]$. For $A \in \mathcal{F}_n$,

$$\int_A Y dP = \int_A E[Y|\mathcal{F}_n]dP = \int_A X_n dP \to \int_A X_\infty dP,$$

by L_1-convergence. So

$$\int_A Y dP = \int_A X_\infty dP,$$

for all $A \in \mathcal{F}_n$, for each n. As the $\mathcal{F}_n$ generate $\mathcal{F}_\infty$, this extends to $A \in \mathcal{F}_\infty$ (by a monotone class argument, or Carathéodory's Extension Theorem). As X_n is $\mathcal{F}_n$-measurable and $\mathcal{F}_n \subset \mathcal{F}_\infty$, X_n is $\mathcal{F}_\infty$-measurable, hence so is its limit X_∞. So

$$X_\infty = E[Y|\mathcal{F}_\infty],$$

by definition of conditional expectation. //

If the index set $\{1, 2, \ldots\}$ of the filtration $(\mathcal{F}_n)$ extends to $\{1, 2, \ldots, \infty\}$ so that $\{X_n : n = 1, 2, \ldots, \infty\}$ is a (sub-)mg w.r.t. this filtration, the (sub-)mg is called *closed*, with *closing* (or *last*) element X_∞.

Theorem. Let (X_n) be a UI submg. Then $\sup_n E[X_n^+] < \infty$, and X_n converges to a limit X_∞ a.s. and in L_1, which closes the submg.

Proof. By UI, $\sup E[|X_n|] < \infty$. So by Doob's Mg Convergence Thm, $X_n \to X_\infty$ a.s. Again by UI, $X_n \to X_\infty$ in L_1.

If $A_n \in \mathcal{F}_n$ and $k \geq n$, $\int_A X_n dP \leq \int_A X_k dP$ as (X_n) is a submg. Let $k \to \infty$: $X_k \to X_\infty$ in L_1 gives $\int_A X_n dP \leq \int_A X_\infty dP$. So by definition of conditional expectation, $X_n \leq E[X_\infty|\mathcal{F}_\infty]$. So X_∞ closes the submg. //

Theorem. X_n is a UI mg iff there exists $Y \in L_1$ with

$$X_n = E[Y|\mathcal{F}_n].$$

Then $X_n \to E[Y|\mathcal{F}_\infty]$ a.s. and in L_1.

Proof. If X is a UI mg, it is closed (by X_∞), by above, and then $X_n \to X_\infty$ a.s. and in L_1; take $Y := X_\infty$.

Conversely, given $Y \in L_1$ and $X_n := E[Y|\mathcal{F}_n]$, (X_n) is a mg, and is UI by above; the convergence follows by Lévy's result above. //