spl2.tex
Lecture 2. 10.10.2011.

2. Classes of sets.

We adopt the usual notational convention: lower case letters for points, or elements of sets; capitals for sets; curly capitals for classes of sets.

We begin with the class $\mathcal{O}$ of *open* sets O. Recall that in Euclidean space – or more generally, a metric space where we have a distance d (generalizing Euclidean distance) – a set O is open if and only if (iff) for each point $x \in O$, all points close enough to x also lie in O. Metric spaces are spaces with a distance, or metric, $d(.,.)$ satisfying
(i) the triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$;
(ii) $d(x, y) = 0$ iff $x = y$.
They were introduced by the French mathematician Maurice FRÉCHET (1878-1973), in his thesis of 1906.
Note. 1. Often the letters G, $\mathcal{G}$ are used instead of O, $\mathcal{O}$ (g for geöffnet, = open in German). Following Schilling [S], we will instead use G, $\mathcal{G}$ for 'generating'.
2. Openness also makes sense more generally, where there may be no distance concept. Here we choose which sets to declare open, subject to the following requirements, which abstract the essence of the above:
(i) the empty set $\emptyset$ and the whole space Ω are open;
(ii) arbitrary unions of open sets are open;
(iii) finite intersections of open sets are open.
A set Ω with such a family of subsets $O \in \mathcal{O}$, $(\Omega, \mathcal{O})$, is called a *topological space*; $\mathcal{O}$ is called the *topology*; the sets O are called the *open sets*. Topology was introduced by the German mathematician Felix HAUSDORFF (1868-1942), in his book *Grundzüge der Mengenlehre* (Foundations of set theory) in 1914.

An open set containing a point x (or more generally, a set containing an open set containing x) is called a *neighbourhood* (nhd for short) of x. A point x is called a *closure point* (or *limit point*) of a set A if every nhd of x meets (= has non-empty intersection with) A. A set is called *closed* iff it contains all its closure points. We write $\mathcal{F}$ for the class of all closed sets F (f for fermé, = closed in French).

These concepts are linked:

Proposition.

A set A is closed (open) iff its complement A^c $(:= \Omega \setminus A)$ is open (closed).

Geometrically, the subject of Topology is about properties preserved under continuous deformation. Analytically (which is what concerns us here), this boils down to a study of openness (or closedness), so here open and closed are diametrical opposites (recall we pass from one to the other by taking complements). By contrast, in Measure Theory one of a set and its complement is as good as the other, so this puts open and closed sets on the same footing.

Recall (Lecture 1) that we talked about finite and countable additivity. We will be handling countable set-theoretic operations constantly; we need some notation for them. If $\mathcal{A}$ is a class of sets A,
(i) the class $\mathcal{A}_\sigma$ is the class obtainable from $\mathcal{A}$ by countable unions;
(ii) the class $\mathcal{A}_\delta$ is the class obtainable from $\mathcal{A}$ by countable intersections
(σ for Summe, = sum (or union), δ for Durschnitt, = intersection (German – Hausdorff's notation)).
Recall also that an arbitrary union of open sets is open, and a finite intersection of open sets is open. A countable intersection of open sets may well be closed – e.g. $\cap_{n=1}^{\infty}(a-1/n, b+1/n) = [a, b]$. So $\mathcal{O}_\sigma = \mathcal{O}$, but $\mathcal{O}_\delta$ is a new class (warning: the usual notation is $\mathcal{G}_\delta$). Similarly, or by taking complements and using the De Morgan laws

$$(A \cup B)^c = A^c \cap B^c, \qquad (A \cap B)^c = A^c \cup B^c,$$

an arbitrary intersection of closed sets is closed, so $\mathcal{F}_\delta = \mathcal{F}$ gives nothing new, but $\mathcal{F}_\sigma$ gives a new class. One can iterate this, and form new and bigger classes

$$\mathcal{O} \subset \mathcal{O}_\delta \subset \mathcal{O}_{\delta\sigma} \subset \mathcal{O}_{\delta\sigma\delta} \subset \ldots,$$

$$\mathcal{F} \subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \subset \mathcal{F}_{\sigma\delta\sigma} \subset \ldots.$$

The example $\cap_{n=1}^{\infty}(a - 1/n, b + 1/n) = [a, b]$ used above generalizes:

$$\mathcal{F} \subset \mathcal{O}_\delta.$$

Similarly (or by taking complements and using De Morgan's Laws),

$$\mathcal{O} \subset \mathcal{F}_\sigma.$$

Taking countable unions of the first and countable intersections of the second,

$$\mathcal{F}_\sigma \subset \mathcal{O}_{\delta\sigma}, \qquad \mathcal{O}_\delta \subset \mathcal{F}_{\sigma\delta}.$$

One can continue in this way: each of the classes in the two increasing sequences of classes above is properly contained in its upper or lower neighbour to the right.

Definition. A *field* (in some books, an *algebra*) is a class $\mathcal{A}$ of subsets of a set Ω such that
(i) $\Omega \in \mathcal{A}$;
(ii) $\mathcal{A}$ is closed under complements: if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$:
(iii) $\mathcal{A}$ is closed under unions: if $A_1,\ A_2 \in \mathcal{A}$, then $A_1 \cup A_2 \in \mathcal{A}$.
A *σ-field* (or *σ-algebra*) is a class $\mathcal{A}$ satisfying (i), (ii) and
(iii*) $\mathcal{A}$ is closed under countable unions: $\mathcal{A}_\sigma = \mathcal{A}$, or equivalently, if $A_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, then

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

Defn. For a class $\mathcal{G}$ of sets, the σ-field *generated* by $\mathcal{G}$ is $\sigma(\mathcal{G})$, the smallest σ-field containing $\mathcal{G}$ – equivalently, the intersection of all σ-fields containing $\mathcal{G}$.
Note. The intersection of two σ-fields, or any number of σ-fields, is again a σ-field, as one may check from the first definition. Hence the two forms in the definition above are indeed equivalent.
Defn. In a topological (or metric) space, the *Borel* σ-field $\mathcal{B}$ is the σ-field generated by the open sets (equivalently, by the closed sets – as σ-fields are closed under complements).

The increasing sequences of classes obtained from the open sets $\mathcal{O}$ and closed sets $\mathcal{F}$ above by taking countable unions and intersections ($._\sigma$ and $._\delta$) are all contained in $\mathcal{B}$, and $\mathcal{B}$ is the smallest σ-field containing them all. So we can think of $\mathcal{B}$ as closing off each sequence at the right-hand end. The resulting sequences of classes, starting from $\mathcal{O}$, $\mathcal{F}$ and ending with $\mathcal{B}$, form the *Borel hierarchy*, so called after the work of Émile BOREL (1871-1956), from his 1893 thesis on.

We quote from Real Analysis that a set O is open on the line iff it is a (finite or) countable disjoint union of open intervals. So the open intervals generate the open sets by countably many set-theoretic operations, and these in turn generate the Borel sets. Combining, the open intervals (a, b) generate the Borel sets on the line. Similarly, so do the closed intervals $[a, b]$ (by taking complements), and the half-open intervals $(a, b]$, $[a, b)$ (by using increasing or decreasing sequences, as in the example above). We use half-open intervals $(a, b]$: the class of finite unions of such intervals is closed, under set-theoretic difference

$$A \setminus B := A \cap B^c = \{x : x \in A, x \notin B\}$$

and union. So each of these four families of intervals forms a family $\mathcal{G}$ which *generates* $\mathcal{B}$: $\sigma(\mathcal{G}) = \mathcal{B}$.

In practice, it is more economical to use a smaller generating family when we can. We can do so in the above by restricting to *rational* end-points a, b (just take rational approximations a_n, b_n increasing or decreasing to a, b as needed – cf. the example above).

Note. The class $\mathcal{G}$ of (any of the four kinds of) intervals with rational end-points is countable. But $\sigma(\mathcal{G}) = \mathcal{B}$ is uncountable. Recall that the union of countably many countable sets is countable (Cantor – below) – but countable set-theoretic operations on a countable family may generate an uncountable family, as here.

Similarly in the plane: the rectangles $(a, b] \times (c, d]$ generate the planar open sets. Similarly in 3 dimensions, the cuboids $(a_1, b_1] \times (a_2, b_2] \times (a_3, b_3]$ generate the open sets, and in d dimensions, so do the cartesian products $\times_{i=1}^{d}(a_i, b_i]$.

The *symmetric difference* of two sets A and B is

$$A \Delta B := (A \cap B^c) \cup (B \cap A^c) = (A \setminus B) \cup (B \setminus A).$$

This is the set of points in exactly one of A and B.

Note. Measure Theory depends on *countable* set-theoretic operations. This depends on the work of Georg CANTOR (1845-1918). First, Cantor showed in 1872 how to construct the reals in terms of Cauchy sequences of rationals. This process is called *completion*, and is quite general – it can be done in any metric space. By contrast, the other way of constructing the reals, also done in 1872 (by Richard DEDEKIND (1831-1916)), in terms of *Dedekind cuts* (or *Dedekind sections*), depends on the total ordering of the reals, so is specific to the reals. Second, Cantor showed in 1873 that the rationals are countable (as the union of countably many countable sets is countable). Third, he showed in 1874 that the reals are uncountable. Finally, in the 1880s he created *transfinite arithmetic* – the theory of infinite cardinal and ordinal numbers. All this created the basis for Measure Theory, and much of 20th century mathematics. All told, Cantor is often regarded as the most influential (or most 'modern') mathematician of the 19th century.