spl21.tex

Lecture 21. 25.11.2011

Construction of BM. It suffices to construct BM for $t \in [0,1]$). This gives $t \in [0,n]$ by dilation, and $t \in [0,\infty)$ by letting $n \to \infty$. First, take $L^2[0,1]$, and any complete orthonormal system (cons) (ϕ_n) on it. Now L^2 is a Hilbert space, under the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx \qquad (\text{or } \int fg),$$

so norm $\|f\| := (\int f^2)^{1/2})$. By Parseval's identity,

$$\int_0^1 fg = \sum_{n=0}^{\infty} \langle f, \phi_n\rangle\langle g, \phi_n\rangle$$

(where convergence of the series on the right is in L^2, or in mean square: $\|f - \sum_0^n \langle f, \phi_k\rangle \phi_k\| \to 0$ as $n \to \infty$). Now take, for $s, t \in [0,1]$,

$$f(x) = I_{[0,s]}(x), \qquad g(x) = I_{[0,t]}(x).$$

Parseval's identity becomes

$$\min(s,t) = \sum_{n=0}^{\infty} \int_0^s \phi_n(x)dx \int_0^t \phi_n(x)dx.$$

Now take (Z_n) independent and identically distributed $N(0,1)$ (recall from II.9, L13 that we can construct these, indeed from one $X \sim U[0,1]$), and write

$$W_t = \sum_{n=0}^{\infty} Z_n \int_0^t \phi_n(x)dx.$$

This is a sum of independent zero-mean random variables. Kolmogorov's theorem on random series says that it converges a.s. if the sum of the variances converges (we quote this). This is $\sum_{n=0}^{\infty} (\int_0^t \phi_n(x)dx)^2$, $= t$ by above. So the series above converges a.s., and by excluding the exceptional null set from our probability space (as we may), everywhere.

The Haar System. Define

$$H(t) = \begin{cases} 1 & \text{on } [0, \frac{1}{2}), \\ -1 & \text{on } [\frac{1}{2}, 1], \\ 0 & \text{else.} \end{cases}$$

Write $H_0(t) \equiv 1$, and for $n \geq 1$, express n in dyadic form as $n = 2^j + k$ for a unique $j = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^j - 1$. Using this notation for n, j, k throughout, write

$$H_n(t) := 2^{j/2} H(2^j t - k)$$

(so H_n has support $[k/2^j, (k+1)/2^j]$). So if m, n $(m \neq n)$ have the same j, $H_m H_n \equiv 0$, while if m, n have different js, one can check that $H_m H_n$ is $2^{(j_1+j_2)/2}$ on half its support, $-2^{(j_1+j_2)/2}$ on the other half, so $\int H_m H_n = 0$. Also H_n^2 is 2^j on $[k/2^j, (k+1)/2^j]$, so $\int H_n^2 = 1$. Combining:

$$\int H_m H_n = \delta_{mn},$$

and (H_n) form an orthonormal system, called the *Haar system*. For completeness: the indicator of any dyadic interval $[k/2^j, (k+1)/2^j]$ is in the linear span of the H_n (difference two consecutive H_ns and scale). Linear combinations of such indicators are dense in $L^2[0,1]$. Combining: the Haar system (H_n) is a complete orthonormal system in $L^2[0,1]$.

The Schauder System. We obtain the *Schauder system* by integrating the Haar system. Consider the triangular function (or 'tent function')

$$\Delta(t) = \begin{cases} 2t & \text{on } [0, \frac{1}{2}), \\ 2(1-t) & \text{on } [\frac{1}{2}, 1], \\ 0 & \text{else.} \end{cases}$$

Write $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$, and define the nth *Schauder function* Δ_n by

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \geq 1).$$

Note that Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval, which is small for n, j large). We see that

$$\int_0^t H(u)du = \frac{1}{2}\Delta(t),$$

and similarly

$$\int_0^t H_n(u)du = \lambda_n \Delta_n(t),$$

where $\lambda_0 = 1$ and for $n \geq 1$,

$$\lambda_n = \frac{1}{2} \times 2^{-j/2} \qquad (n = 2^j + k \geq 1).$$

The Schauder system (Δ_n) is again a cons on $L^2[0,1]$.

Theorem (Paley-Wiener-Zygmund, 1933). For $(Z_n)_0^\infty$ independent $N(0,1)$ random variables, λ_n, Δ_n as above,

$$W_t := \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

converges uniformly on $[0,1]$, a.s. The process $W = (W_t : t \in [0,1])$ is Brownian motion.

Lemma. For Z_n independent $N(0,1)$,

$$|Z_n| \leq C\sqrt{\log n} \qquad \forall n \geq 2,$$

for some random variable $C < \infty$ a.s.

Proof of the Lemma. For $x > 1$,

$$P(|Z_n| \geq x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du \leq \sqrt{2/\pi} \int_x^\infty u e^{-\frac{u^2}{2}} du = \sqrt{2/\pi} e^{-\frac{x^2}{2}}.$$

So for any $a > 1$,

$$P(|Z_n| > \sqrt{2a \log n}) \leq \sqrt{2/\pi} \exp\{-a \log n\} = \sqrt{2/\pi} n^{-a}.$$

Since $\sum n^{-a} < \infty$ for $a > 1$, the Borel-Cantelli lemma gives

$$P(|Z_n| > \sqrt{2a \log n} \text{ for infinitely many } n) = 0.$$

So

$$C := \sup_{n \geq 2} \frac{|Z_n|}{\sqrt{\log n}} < \infty \qquad a.s.$$

Proof of the Theorem.
1. Convergence. Choose J and $M \geq 2^J$; then

$$\sum_{n=M}^{\infty} \lambda_n |Z_n| \Delta_n(t) \leq C \sum_{M}^{\infty} \lambda_n \sqrt{\log n} \Delta_n(t).$$

The right is majorized by

$$C\sum_{J}^{\infty}\sum_{k=0}^{2^j-1}\frac{1}{2}2^{-j/2}\sqrt{j+1}\Delta_{2^j+k}(t)$$

(perhaps including some extra terms at the beginning, using $n = 2^j + k < 2^{j+1}, \log n \le (j+1)\log 2$, and $\Delta_n(.) \ge 0$, so the series is absolutely convergent). In the inner sum, only one term is non-zero (t can belong to only one dyadic interval $[k/2^j, (k+1)/2^j)$), and each $\Delta_n(t) \in [0,1]$. So

$$LHS \le C\sum_{j=J}^{\infty}\frac{1}{2}2^{-j/2}\sqrt{j+1} \qquad \forall t \in [0,1],$$

and this tends to 0 as $J \to \infty$, so as $M \to \infty$. So the series $\sum \lambda_n Z_n \Delta_n(t)$ is absolutely and uniformly convergent, a.s. Since continuity is preserved under uniform convergence and each $\Delta_n(t)$ (so each partial sum) is continuous, W_t is continuous in t.

2. Covariance. By absolute convergence and Fubini's theorem,

$$E(W_t) = E\left(\sum_0^{\infty}\lambda_n Z_n \Delta_n(t)\right) = \sum \lambda_n \Delta_n(t) E(Z_n) = \sum 0 = 0.$$

So the covariance is

$$E(W_s W_t) = E\left[\sum_m Z_m \int_0^s \phi_m \times \sum_n Z_n \int_0^t \phi_n\right] = \sum_{m,n} E[Z_m Z_n]\int_0^s \phi_m \int_0^t \phi_n,$$

or as $E[Z_m Z_n] = \delta_{mn}$,

$$\sum_n \int_0^s \phi_m \int_0^t \phi_n = \min(s,t),$$

by the Parseval calculation above.