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**Lecture 24. 2.12.2011**

*Exponential Distribution*

A random variable $T$ is said to have an exponential distribution with rate $\lambda$, or $T = E(\lambda)$ if

$$P(T \leq t) = 1 - e^{-\lambda t} \qquad \text{for all } \; t \geq 0.$$

Recall $E(T) = 1/\lambda$ and $var(T) = 1/\lambda^2$. Further important properties are:
(i) Exponentially distributed random variables possess the 'lack of memory' property: $P(T > s + t | T > t) = P(T > s)$.
(ii) Let $T_1, T_2, \ldots T_n$ be independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ resp. Then $\min\{T_1, T_2, \ldots, T_n\}$ is exponentially distributed with rate $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.
(iii) Let $T_1, T_2, \ldots T_n$ be independent exponentially distributed random variables with parameter $\lambda$. Then $G_n = T_1 + T_2 + \ldots + T_n$ has a $Gamma(n, \lambda)$ distribution. That is, its density is

$$P(G_n = t) = \lambda e^{-\lambda t} (\lambda t)^{n-1}/(n-1)! \qquad \text{for } \; t \geq 0.$$

*The Poisson Process*

*Definition.* Let $t_1, t_2, \ldots t_n$ be independent exponential $E(\lambda)$ random variables. Let $T_n = t_1, + \ldots + t_n$ for $n \geq 1$, $T_0 = 0$, and define $N(s) = \max\{n : T_n \leq s\}$.
*Interpretation:* Think of $t_i$ as the time between arrivals of events, then $T_n$ is the arrival time of the $n$th event and $N(s)$ the number of arrivals by time $s$. Then $N(s)$ has a Poisson distribution with mean $\lambda s$. The Poisson process can also be characterised via

**Theorem**. If $\{N(s), s \geq 0\}$ is a Poisson process, then
(i) $N(0) = 0$,
(ii) $N(t + s) - N(s) = Poisson(\lambda t)$, and
(iii) $N(t)$ has independent increments.
Conversely, if (i),(ii) and (iii) hold, then $\{N(s), s \geq 0\}$ is a Poisson process.

The above characterization can be used to extend the definition of the Poisson process to include time-dependent intensities. We say that $\{N(s), s \geq 0\}$ is a *Poisson process* with *rate* $\lambda(r)$ if

(i) $N(0) = 0$,
(ii) $N(t+s) - N(s)$ is Poisson with mean $\int_s^t \lambda(r)dr$, and
(iii) $N(t)$ has independent increments.

*Compound Poisson Processes*

We now associate i.i.d. random variables $Y_i$ with each arrival and consider

$$S(t) = Y_1 + \ldots + Y_{N(t)}, \qquad S(t) = 0 \text{ if } N(t) = 0.$$

**Theorem**. Let $(Y_i)$ be i.i.d. and $N$ be an independent nonnegative integer random variable, and $S$ as above.
(i) If $E(N) < \infty$, then $E(S) = EX(N).E(Y_1)$.
(ii) If $E(N^2) < \infty$, then $var(S) = E(N).var(Y_1) + var(N)(E(Y_1))^2$.
(iii) If $N = N(t)$ is Poisson$(\lambda t)$, then $var(S) = t\lambda(E(Y_1))^2$.

A typical application in the insurance context is a Poisson model of claim arrival with random claim sizes.

*Renewal Processes*

Suppose we use components – light-bulbs, say – whose lifetimes $X_1, X_2, \ldots$ are independent, all with law $F$ on $(0, \infty$. The first component is installed new, used until failure, then replaced, and we continue in this way. Write

$$S_n := \sum_1^n X_i, \qquad N_t := \max\{k : S_k < t\}.$$

Then $N = (N_t : t \geq 0)$ is called the *renewal process* generated by $F$; it is a *counting process*, counting the number of failures seen by time $t$.

The law $F$ has the *lack-of-memory property* iff the components show no aging – that is, if a component still in use behaves as if new. The condition for this is

$$P(X > s + t | X > s) = P(X > t) \qquad (s, t > 0),$$

or

$$P(X > s + t) = P(X > s)P(X > t).$$

Writing $\overline{F}(x) := 1 - F(x)$ $(x \geq 0)$ for the *tail* of $F$, this says that

$$\overline{F}(s + t) = \overline{F}(s)\overline{F}(t) \qquad (s, t \geq 0).$$

Obvious solutions are

$$\overline{F}(t) = e^{-\lambda t}, \qquad F(t) = 1 - e^{-\lambda t}$$

for some $\lambda > 0$ – the exponential law $E(\lambda)$. Now

$$f(s+t) = f(s)f(t) \qquad (s, t \geq 0)$$

is a 'functional equation' – the *Cauchy functional equation* – and it turns out that these are the *only* solutions, subject to minimal regularity (such as one-sided boundedness, as here – even on an interval of arbitrarily small length!).

So the exponential laws $E(\lambda)$ are *characterized* by the lack-of-memory property. Also, the lack-of-memory property corresponds in the renewal context to the *Markov property*. The renewal process generated by $E(\lambda)$ is called the *Poisson (point) process* with *rate* $\lambda$, $Ppp(\lambda)$. So: among renewal processes, the only Markov processes are the Poisson processes. We meet Lévy processes below: among renewal processes, the only Lévy processes are the Poisson processes.

It is the lack of memory property of the exponential distribution that (since the inter-arrival times of the Poisson process are exponentially distributed) makes the Poisson process the basic model for events occurring 'out of the blue'.

## 8. Lévy Processes

*Distributions; The Lévy-Khintchine Formula*

A distribution is *infinitely divisible* (id) if for each $n$ its CF $\phi$ is the $n$th power $\phi_n^n$ of a CF $\phi_n$. The class of infinitely divisible laws is written $ID$. The form of the general infinitely-divisible distribution was studied in the 1930s by several people (including Kolmogorov and de Finetti). The final result, due to Lévy and Khintchine, is expressed in CF language – indeed, cannot be expressed otherwise. The Lévy-Khintchine formula below is a static result; its dynamic counterpart involves *Lévy processes* (stochastic processes with stationary independent increments). We return to these in IV.6 [L30] in connection with stochastic calculus.

To describe the CF of the general i.d. law, we need three components: (i) a real $a$ (called the *drift*, or deterministic drift), (ii) a non-negative $\sigma$ (called the *diffusion coefficient*, or normal component, or Gaussian component), (iii) a (positive) measure $\mu$ on $\mathbf{R}$ (or $\mathbf{R} \setminus \{0\}$) for which

$$\int_{-\infty}^{\infty} \min(1, |x|^2)\mu(dx) < \infty,$$

that is,

$$\int_{|x|<1} |x|^2 \mu(dx) < \infty, \qquad \int_{|x|\geq 1} \mu(dx) < \infty,$$

called the *Lévy measure*. The result is

**Theorem (Lévy-Khintchine Formula)**. A function $\phi$ is the characteristic function of an infinitely divisible distribution iff it has the form

$$\phi(u) = \exp\{-\Psi(u)\} \qquad (u \in \mathbf{R}),$$

where

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI_{(-1,1)}(x)\mu(dx) \qquad (L-K)$$

for some real $a, \sigma \geq 0$ and Lévy measure $\mu$.

*Examples.* These include the normal, Poisson, compound Poisson and Cauchy laws (see below under 'Stability' for Cauchy).
*The Central Limit Problem.* In the CLT of II.7 [L12], we found that the limits we can get from an iid sequence by centring and scaling (subtracting means and dividing by variances there) were normal. The classical *central limit problem* generalizes this to sums $\Sigma_k X_{nk}$ from a 'triangular array' ($1 \leq k \leq k_n < \infty$, $n = 1, 2, \ldots$; $X_{nk}$ independent as $k$ varies for fixed $n$). It turns out that the class of possible limit laws is exactly the class $ID$ of infinitely divisible laws in $(L-K)$.
*Self-decomposability.* Recall that if, in the central limit problem, we restrict from (two-suffix) triangular arrays $(X_{nk})$ to (one-suffix) sequences $(X_n)$, we come to a subclass of the infinite-divisible laws $I$, called the class of self-decomposable laws $SD: \ SD \subset I$.
*Stability.* If we further restrict to iid sequences $X_n$, we get the class $S$ of *stable* laws:

$$S \subset SD \subset I.$$

To within location and scale, these are described by two parameters, the *index* $\alpha \in (0, 2]$ and the *skewness parameter* $\beta \in [-1, 1]$; $\alpha = 2$ gives the normal law and $\beta = 0$ gives symmetry. The (symmetric) *Cauchy* law is the case $\alpha = 1, \beta = 0$; density $1/(\pi(1 + x^2))$.