spl30.tex
Lecture 30. 16.12.2011.

The Ornstein-Uhlenbeck Process. The most important example of a stochastic differential equation for us is that for geometric Brownian motion. We close here with another example.

Consider a model of the velocity $V(t)$ of a particle at time t ($V(0) = v_0$), moving through a fluid or gas, which exerts a force on the particle consisting of:
(i) a frictional drag, assumed proportional to the velocity,
(ii) a noise term resulting from the random bombardment of the particle by the molecules of the surrounding fluid or gas.
The basic model for processes of this type is given by the (linear) stochastic differential equation

$$dV = -\beta V dt + \sigma dW,$$

whose solution is called the *Ornstein-Uhlenbeck* (velocity) process with relaxation time $1/\beta$ and diffusion coefficient $D := \frac{1}{2}\sigma^2/\beta^2$. It is a stationary Gaussian Markov process (not stationary-increments Gaussian Markov like Brownian motion), whose limiting (ergodic) distribution is $N(0, \beta D)$ (this is the classical Maxwell-Boltzmann distribution of statistical mechanics) and whose limiting correlation function is $e^{-\beta|\cdot|}$.

If we integrate the Ornstein-Uhlenbeck velocity process to get the Ornstein-Uhlenbeck displacement process, we lose the Markov property (though the process is still Gaussian). Being non-Markov, the resulting process is much more difficult to analyze.

The Ornstein-Uhlenbeck process is important in many areas, including:
(i) statistical mechanics, where it originated,
(ii) mathematical finance, where it appears in the *Vasicek model* for the term-structure of interest-rates.

We solve the stochastic differential equation. We know that $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta C e^{-\beta t} dt + e^{-\beta t} dC = -\beta V dt + e^{-\beta t} dC,$$

so V is a solution of (OU) if $e^{-\beta t} dC = \sigma dW$, $dC = \sigma e^{\beta t} dW$, $C = c + \int_0^t e^{\beta u} dW$. So with initial velocity v_0,

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u.$$

This approach to solving linear SDEs can be generalized.

6. Semi-martingales.

The martingale concept, though crucial, is a little too restrictive, and one needs to generalize it. We will be brief here. First, a local martingale $M = (M(t))$ is a process such that, for some sequence of stopping times $S_n \to \infty$, each stopped process $M^{(n)} = (M(t \wedge S_n))$ is a martingale. This localization idea can be applied elsewhere: a process $(A(t))$ (adapted to our filtration, understood) is locally of finite variation if each $(A(t \wedge S_n))$ is of finite variation for some sequence of stopping times $S_n \to \infty$. A *semi-martingale* (Meyer, 1976) is a process $(X(t))$ expressible as

$$X(t) = M(t) + A(t)$$

with $(M(t))$ a local martingale and $(A(t))$ locally of finite variation (the concept is due to Meyer).

Lévy Processes as Semi-martingales. The Gaussian component $X^{(1)}$ is a martingale; so too is the compensated sum of (small) jumps process $X^{(3)}$, while the sum of large jumps process $X^{(2)}$ is (locally) of finite variation, being compound Poisson. Thus a Lévy process $X = X^{(1)} + X^{(2)} + X^{(3)}$ is a semi-martingale. Indeed, Lévy processes are the prototypes, and motivating examples, of semi-martingales. The natural domain of stochastic integration is predictable integrands and semi-martingale integrators. Thus, stochastic integration works with a general Lévy process as integrator. Here, however, the theory simplifies considerably.

Previsible (= Predictable) Processes. The crucial difference between left-continuous (e.g., càglàd) functions and right-continuous (e.g., càdlàg) ones is that with *left*- continuity, one can 'predict' the value at t - 'see it coming' - knowing the values before t.

We write $\mathcal{P}$, called the *predictable* (or previsible) σ-algebra, for the σ-algebra on $\mathbf{R}_+ \times \Omega$ ($\mathbf{R}_+$ for time $t \geq 0$, Ω for randomness $\omega \in \Omega$ - we need both for a stochastic process $X = (X(t, \omega))$) for the σ-field generated by (= smallest σ-field containing) the adapted càglàd processes. (We shall almost always be dealing with adapted processes, so the operative thing here is the *left*-continuity.) We also write $X \in \mathcal{P}$ as shorthand for 'the process X is $\mathcal{P}$-measurable', and $X \in b\mathcal{P}$ if also X is bounded.

Predictability and Semi-Martingales. Let us confess here why we need to introduce the last two concepts. One can develop a theory of *stochastic integrals*, $\int_0^t H(s,\omega)dM(s,\omega)$ or $\int_0^t H_s dM_s$, where H, M are stochastic processes and the *integrator* M is a *semimartingale*, the *integrand* H is *previsible* (and bounded, or L_2, or whatever). This can be done; see e.g. [P] for details. More: this theory is the *most general* theory of stochastic integration *possible*, if one demands even reasonably good properties (appropriate behaviour under passage to the limit, for example). For emphasis:

Integrands: previsible; Integrators: semimartingales.

Prototype: H is left-continuous (and bounded, or L_2, etc.); M is Brownian motion.
Economic Interpretation. Think of the integrator M as, e.g., a stock-price process. The increments over $[t, t+u]$ ($u > 0$, small) represent 'new information'. Think of the integrand H as the amount of stock held. The investor has no advance warning of the price change $M_{t+dt} - M_t$ over the immediate future $[t, t + dt]$, but has to commit himself on the basis of what he knows already. So H needs to be predictable at H before t (e.g., left- continuity will do), hence predictability of integrands. By contrast, $M_{t+dt} - M_t$ represents new price-sensitive information, or 'driving noise'. The value process of the portfolio is the limit of sums of terms such as $H_{t-}(M_{t+dt} - M_t)$, the stochastic integral $\int_0^t H_s dM_s$. This is the continuous-time analogue of the *martingale transform* in discrete time (III.2).

Poisson Stochastic Calculus. Recall that the prototypes of Lévy processes are Brownian motion and the Poisson process, also that the essence of Itô calculus for BM is $(dW_t)^2 = dt$. Now the Poisson process N is a point process with jumps of size 1, so $(dN_t)^2 = dN_t$ (both sides are 1 at a jump and 0 elsewhere). This suggests that a Poisson-based stochastic calculus can be developed, and indeed it can.

Lévy stochastic calculus. With both Brownian and Poissonian calculus to hand, this suggests that stochastic calculus for Lévy processes can be developed – and indeed it can. For, Lévy processes are semimartingales, and we saw above that stochastic calculus has as its natural domain that of predictable integrands and semimartingale integrators. The resulting Lévy calculus is very flexible and useful, but we cannot develop it here. It extends Black-Scholes theory to allow prices to have *jumps*, which they do in reality if looked at closely enough.

Lévy finance. We close with some comments on the use of Lévy processes for modelling in mathematical finance. There are three main objections to the use of Brownian-based models, as in Black-Scholes theory.
(i) Gaussian distributions are symmetric, and have extremely thin tails. Real financial data show skew, and have much fatter tails than Gaussian. For example, with return distributions on stock, the tail behaviour depends on the length of the return interval. For monthly returns, say, returns are approximately Gaussian. This is because of *aggregational Gaussianity*: the Central Limit Theorem applies. The rule of thumb is that 16 trading days suffice here. High-frequency ('tick') data typically gives heavy tails – tails decreasing like a power; daily returns are intermediate (e.g., hyperbolic distributions).
(ii) Brownian models are *complete* (see L29, re the Brownian Martingale Representation Theorem). Real markets are *incomplete*. One can see this in, e.g., the *bid-ask spread* – real prices are not unique, but fill an interval.
(iii) Brownian motion is continuous, but real prices *jump*. This is partly because prices are quoted in terms of money, which is quantised. Also, the very act of trading shifts prices, as it affects the current balance of supply and demand. In Black-Scholes theory, one assumes that financial agents are price takers and not price makers – true to a good approximation for small traders (or small trades), but not for large ones. Where there is no trading, there is no price. Where there is trading, there are prices rather than a price. Take, for instance, the price evolution of a heavily traded (and so highly liquid) stock under normal market conditions. There will be very many individually small trades, resulting in what is called *jitter*. Lévy processes of *infinite activity* – infinitely many jumps in finite time – are well suited to modelling such things. What was once pure Probability Theory for its own sake has now become an everyday modelling tool for the financial practitioner.

The mathematics of markets under crisis conditions is of course very interesting and topical, but we cannot develop it here.

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