spprob2.tex

Problems 2. 21.10.2011

Q1. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$, show that
(i) if $A_n \uparrow$ i.e. $A_n \subset A_{n+1} \subset \ldots$), then

$$\mu(A_n) \uparrow \mu(\cup A_n)$$

('continuity of μ from below');
(ii) if $A_n \downarrow$ (i.e. $A_n \supset A_{n+1} \supset \ldots$) and some $\mu(A_N) < \infty$

$$\mu(A_n) \downarrow \mu(\cap A_n)$$

('continuity of μ from above').

Q2. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$, show that
(i)

$$\mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n);$$

(ii) if $\mu(\cup_{k \geq N} A_k) < \infty$ for some N,

$$\mu(\text{limsup } A_n) \geq \text{limsup } \mu(A_n).$$

Q3. If $(\Omega, \mathcal{A}, \mu)$ is a measure space and $A_n \in \mathcal{A}$ is a convergent sequence of sets with $\mu(\cup A_n) < \infty$, show that

$$\mu(\lim A_n) = \lim \mu(A_n).$$

Q4. Show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

exists as in *improper Riemann integral* (the left is defined as the limit of $\int_0^N$ ... as $N \uparrow \infty$), but not as a Lebesgue integral.

Deduce that, although Lebesgue integration is more general than Riemann integration, it is not more general than improper Riemann integration.

NHB