spprob8.tex

Problems 8. 2.12.2011

Q1. (i) If $M = (M_t)$ is a mg, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submg.
(ii) If in (i) M is a submg, and ϕ is also non-decreasing on the range of M, show that again $\phi(M)$ is a submg.

Q2. (i) Deduce from Q1 that for $B = (B_t)$ BM, $B^2 = (B_t^2)$ is a submg. Find the increasing process in its Doob-Meyer decomposition. Deduce that BM has quadratic variation t.
(ii) Show that for $p \geq 1$ $|B|^p$ is a submg.
(iii) Show that B^+ is a submg.

Q3 *(Doob's Submartingale Inequality).* For X a submg, $c \geq 0$, show that

$$cP(\max_{k=1,\dots,n} X_k \geq c) \leq E[X_n I(\max_{k=1,\dots,n} X_k \geq c)] \leq E[X_n].$$

Q4 *Doob's Submartingale Convergence Theorem.* Deduce this result from Q3 and Doob's Upcrossing Inequality.

Q5 *(Second Borel-Cantelli Lemma under Pairwise Independence).* Show that the Second Borel-Cantelli Lemma continues to hold with independence weakened to pairwise independence. [This result is due to Etemadi; there is a proof in [S], Th. 18.9, p.198-9. It uses Tchebycheff's inequality, and the fact that the Bernoulli distribution $B(p)$ with parameter $p \in [0, 1]$ has mean p and variance $p(1 - p)$ (or pq with $q := 1 - p$)].

Q6 *Etemadi's SLLN for pairwise independence.* Use Q5 to extend Etemadi's proof of the SLLN in lectures to pairwise independence.

NHB