spprob9.tex

Problems 9. 10.12.2010

Q1. *First-passage time process of Brownian motion.* For $t \geq 0$ and B BM, write

$$\tau_t := \min\{u : B_u \geq t\}$$

(or $\min\{u : B_u = t\}$ as BM is continuous). So $B(\tau_t) = t$.
(i) Show that for fixed $s \geq 0$, $M_t := \exp\{sB_t - \frac{1}{2}ts^2\}$ is a mg.
(ii) By considering the first-passage times of BM to levels t and $t + u$, show that the process $\tau = (\tau_t)$ is a non-decreasing Lévy process (a subordinator).
(iii) By considering the bounded stopping times $T_n := \min(n, \tau_t)$ and Doob's Stopping Time Principle in continuous time (which you may quote) and letting $n \to \infty$, or otherwise, show that

$$E \exp\{-s\tau_t\} = e^{-t\sqrt{2s}}.$$

(iv) Show that for $c > 0$,

$$\tau_t =_d \tau_{ct}/c^2$$

(so τ is *stable* of index 1/2 – the stable subordinator of index 1/2.

Q2. Show that τ_1 has density

$$f(x) := \frac{1}{\sqrt{2\pi}}.\frac{1}{x^{3/2}}.\exp(-\frac{1}{2x}).$$

[Let f have Laplace transform $\phi(s) := \int_0^\infty e^{-sx} f(x)dx$. Find $\phi'(s)$, and show, by the change of variable

$$x = \frac{1}{2su}, \qquad \text{so} \qquad sx = \frac{1}{2u}, \qquad \frac{1}{2x} = su,$$

that ϕ satisfies the ODE $\phi'(s)/\phi(s) = -1/\sqrt{2s}$.]
(This result is due to Lévy; it is, with BM and the Cauchy distribution, one of the few explicit formulae for a stable density.)

Q3. If $X, X_1, \ldots, X_n$ are independent with the stable-1/2 density in Q2, show that $(X_1 + \ldots + X_n)/n^2 =_d X$. Why does this not contradict the SLLN?

NHB