spsoln10.tex

Solutions 10. 17.12.2010

Q1. (i)

$$\begin{aligned}\psi(t)=E[e^{itY}] &= E[\exp\{it(X_1+\ldots+X_N)\}]\\ &= \sum_n E[\exp\{it(X_1+\ldots+X_N)\}|N=n].P(N=n)\\ &= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1+\ldots+X_n)\}]\\ &= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n\\ &= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n\\ &= \exp\{-\lambda(1-\phi(t))\}.\end{aligned}$$

Differentiate:

$$\psi'(t)=\psi(t).\lambda\phi'(t),$$

$$\psi''(t)=\psi'(t).\lambda\phi'(t)+\psi(t).\lambda\phi''(t).$$

As $\phi(t)=E[e^{itX}]$, $\phi'(t)=E[iXe^{itX}]$, $\phi''(t)=E[-X^2e^{itX}]$. So ($\phi(0)=1$ and) $\phi'(0)=i\mu$, $\phi''(0)=-E[X^2]$,

$$\psi'(0)=\lambda\phi'(0)=\lambda.i\mu,$$

and as also $\psi'(0)=iEY$, this gives $EY=\lambda\mu$. Similarly,

$$\psi''(0)=i\lambda\mu.i\lambda\mu+\lambda\phi''(0)=-\lambda^2\mu^2-\lambda E[X^2],$$

and also ($\psi(0)=1$, $\psi'(0)=i\lambda\mu$ and) $\psi''(0)=-E[Y^2]$. So

$$var\ Y=E[Y^2]-[EY]^2=\lambda^2\mu^2+\lambda E[X^2]-\lambda^2\mu^2=\lambda E[X^2].$$

(ii) Given N, $Y=X_1+\ldots+X_N$ has mean $NEX=N\mu$ and variance $N\ var\ X=N\sigma^2$. As N is Poisson with parameter λ, N has mean λ and variance λ. So by the Conditional Mean Formula,

$$EY=E[E(Y|N)]=E[N\mu]=\lambda\mu.$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2].$$

Q2 . (i). Write $f(B,t) := (B^2 - t)^2$. By Itô's formula,

$$df = f_B dB + f_t dt + \frac{1}{2}[f_{BB}(dB)^2 + 2f_{Bt}dBdt + f_{tt}(dt)^2].$$

In the [...] on RHS, $(dB)^2 = dt$, $dBdt = 0$, $(dt)^2 = 0$. Also

$$f_B = 2.2B(B^2 - t), \qquad f_t = -2(B^2 - t), \qquad f_{BB} = 4(B^2 - t) + 4B.2B = 12B^2 - 4t.$$

So

$$df = 4B(B^2 - t)dB - 2(B^2 - t)dt + (6B^2 - 2t)dt = 4B(B^2 - t)dB + 4B^2 dt.$$

As $M = f - 4\int_0^t B_s^2 ds$,

$$dM = df - 4B_t^2 dt = 4B(B^2 - t)dB :$$

$$M_t = 4\int_0^t B_s(B_s^2 - s)dB_s.$$

The Itô integral on the RHS is a continuous local martingale starting from 0. Now $B_t =_d t^{1/2}.Z$ where Z is $N(0,1)$. As Z has all moments finite, each $E[B_t^n]$ is a polynomial in t. So the integrand $h = h(B_t, t)$ on RHS satisfies the integrability condition $\int_0^t E[h_s^2]ds < \infty$ for all t. So the RHS is a (true) continuous mg starting from 0.

(ii). With $[M] = ([M_t])$ the quadratic variation of M,

$$d[M]_t = (dM)_t^2; \qquad dM_t = 4B_t(B_t^2 - t)dB_t.$$

So

$$d[M]_t = 16B_t^2(B_t^2 - t)^2(dB_t)^2 = 16B_t^2(B_t^2 - t)^2 dt :$$

$$[M]_t = 16\int_0^t B_s^2(B_s^2 - s)^2 ds.$$

Q3. (i) V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u} dW_u = \int_0^t e^{\beta u} E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u)^2] = \sigma^2 \int_0^t (e^{-\beta t + \beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t} \int_0^t e^{-2\beta u} du = \sigma^2 e^{-2\beta t} [e^{2\beta t} - 1]/(2\beta) = \sigma^2 [1 - e^{-2\beta t}]/(2\beta).$$

So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$.
(ii) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which (subtracting off $v_0 e^{-\beta t}$ as we may) is

$$\sigma^2 E[e^{-\beta t} \int_0^t e^{\beta v} dW_v . e^{-\beta(t+u)} (\int_0^t + \int_t^{t+u}) e^{\beta w} dW_w].$$

By independence of Brownian increments, the $\int_t^{t+u}$ term contributes 0, leaving as before

$$cov(V_t, V_{t+u}) = \sigma^2 e^{-\beta u} [1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty).$$

(iii) The process V is Markov (a diffusion), being the solution of the SDE (OU). V is Gaussian, as it is obtained from the Gaussian process W by linear operations.
Note. The limiting distribution $N(0, \sigma^2/(2\beta))$ is the *Maxwell-Boltzmann distribution* of Statistical Mechanics. The limit process is stationary Gaussian Markov, in contrast to Brownian motion, which is Gaussian Markov with stationary increments.

NHB