spsoln2.tex

Solutions 2. 28.10.2011

Q1. (i). Write $B_n := A_n \setminus A_{n+1}$. Then the B_n are disjoint, $A_n = \cup_1^n B_k$ and $\cup A_n = \cup B_n$, so

$$\mu(A_n) = \mu(\cup_1^n B_k) = \sum_1^n \mu(B_k) \uparrow \mu(\sum_1^\infty \mu(B_k).$$

But as μ is a measure,

$$\mu(\cup_1^\infty A_n) = \mu(\cup_1^\infty B_k) = \sum_1^\infty \mu(B_k).$$

So

$$\mu(A_n) \uparrow \mu(\cup A_n).$$

(ii) As $A_n \downarrow$ and $\mu(A_N) < \infty$: for $n \geq N$, $A_n \subset A_N$ and $(A_N \setminus A_n) \uparrow$. So by (i),

$$\mu(A_N \setminus A_n) = \mu(A_N) - \mu(A_n) \uparrow \mu(\cup_{n \geq N} A_N \setminus A_n)$$
$$= \mu(A_N \setminus \cap_{n \geq N} A_n) = \mu(A_N) - \mu(\cap_{n \geq N} A_n).$$

So

$$\mu(A_n) \downarrow \mu(\cap A_n).$$

Q2. Let $B_n := \cap_{k \geq n} A_k$. Then $B_n \subset A_n$, so $\mu(B_n) \leq \mu(A_n)$, liminf $\mu(B_n) \leq$ liminf $\mu(A_n)$. But $B_n \uparrow$, so by Q1(i),

$$\text{liminf } \mu(B_n) = \lim \mu(B_n) = \mu(\cup B_n) = \mu(\text{liminf } A_n).$$

Combining,

$$\mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n), \quad //$$

giving (i). Part (ii) follows similarly from Q1(ii), or by taking complements of (i) w.r.t. $\cup_{k \geq N} A_k$. //

Q3. By Q2(i), (ii),

$$\mu(\lim A_n) = \mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n) \leq \text{limsup } \mu(A_n) \leq \mu(\text{limsup } A_n) = \mu(\lim A_n).$$

Q4. We use Cauchy's Theorem (see e.g. Lecture 27, M2PM3, link on my homepage). To prove:

$$I := \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Take $f(z) = e^{iz}/z$. This has a pole at the origin, which we must exclude from the semi-circular contour we would use as above by a semi-circular indentation round the origin. Take γ the union of γ_1, the semi-circle centre 0 and radius $\epsilon > 0$ in the upper half-plane (clockwise), $\gamma_2 := [\epsilon, R]$, γ_3 the semi-circle radius R in the upper half-plane (anticlockwise) and $\gamma_4 := [-R, -\epsilon]$. By Cauchy's Theorem, $\int_\gamma = 0$. So for $\delta > 0$,

$$\left|\int_{\gamma_3} f\right| = \left|\int_0^\pi \frac{e^{i(R\cos\theta + iR\sin\theta)}}{Re^{i\theta}} \cdot iRe^{i\theta}\, d\theta\right| \le \int_0^\pi e^{-R\sin\theta}\, d\theta = \int_0^\delta + \int_\delta^{\pi-\delta} + \int_{\pi-\delta}^\pi$$

$$\le \delta + \delta + e^{-R\sin\theta}(\pi - 2\delta) : \qquad \limsup_{R\to\infty} \left|\int_{\gamma_3} f\right| \le 2\delta.$$

So as $\delta > 0$ is arbitrarily small: RHS $= 0$. So $\int_{\gamma_3} f \to 0$ $(R \to \infty)$.

$$\int_{\gamma_1} f = \int_0^\pi e^{i\epsilon(\cos\theta + i\sin\theta)} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta = i\int_0^\pi (1 + O(\epsilon))\, d\theta = i\pi + O(\epsilon) \to i\pi \quad (\epsilon \to 0).$$

Also $\int_{I_2} f = \int_{I_4} f$ as $(\sin\ x)/x$ is even, so

$$\int_{I_2} f + \int_{I_4} f = 2\int_2 f \to 2iI \qquad (R \to \infty, \epsilon \downarrow 0).$$

Combining, $I \to \pi/2$ as $R \to \infty$, $\epsilon \downarrow 0$. So the integral exists as an improper Riemann integral, as required.

But the integral does not exists as a Lebesgue integral. If it did, since the Lebesgue integral is an absolute integral, $\int_0^\infty \frac{|\sin x|}{x}\, dx$ would exist also – i.e., would be finite. But $|\sin x| \ge 1/2$ (say) over part of its period $[0, 2\pi]$, A say. Writing A_n for $A + 2\pi n$,

$$\int_0^\infty \frac{|\sin x|}{x} dx = \sum_0^\infty \int_{2n\pi}^{2(n+1)\pi} \ldots \ge \sum_n \int_{A_n} \ldots \ge \sum_n \frac{1}{2}\int_{A_n} dx/x.$$

The series on the right diverges by comparison with the harmonic series $\sum_1^\infty 1/n$.

NHB