

Solutions 5. 18.11.2010

Q1. Take $f(z) := e^{-z^2/2}$. This is entire (has no singularities). So for any contour γ , $\int_{\gamma} f = 0$, by Cauchy's Residue Theorem (or, use Cauchy's Theorem). Take γ the rectangle with vertices R , $R + iy$, $-R + iy$, $-R$, with sides γ_1 the interval $[-R, R]$, γ_2 the vertical line from R to $R + iy$, γ_3 the horizontal line from $R + iy$ to $-R + iy$, γ_4 the vertical line from $-R + iy$ to $-R$. So $\sum_1^4 \int_{\gamma_i} f = 0$. On γ_2, γ_4 : $z = \pm R + iuy$ ($0 \leq u \leq 1$),

$$f(z) = \exp\{-(\pm R + iuy)^2/2\} = e^{-R^2/2} e^{u^2 y^2/2} e^{\pm iRuy} \rightarrow 0 \quad (R \rightarrow \infty),$$

as $|e^{\pm iRuy}| = 1$. So $\int_{\gamma_2} f \rightarrow 0$, $\int_{\gamma_4} f \rightarrow 0$ ($R \rightarrow \infty$). Also $\int_{\gamma_1} f \rightarrow \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ as $R \rightarrow \infty$). Combining,

$$\int_{\gamma_3} f \rightarrow \int_{\infty}^{-\infty} e^{-x^2/2} \cdot e^{y^2/2} \cdot e^{-ixy} dx = -\sqrt{2\pi} \quad (R \rightarrow \infty).$$

So (dividing by $\sqrt{2\pi}$ and by $e^{y^2/2}$, and reversing the direction of integration)

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot e^{-ixy} dx = e^{-y^2/2}.$$

The RHS is real, so the LHS is real. Take complex conjugates:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \cdot e^{ixy} dx = e^{-y^2/2}.$$

This gives the characteristic function (CF) of the standard normal density $\phi(x) := e^{-x^2/2}/\sqrt{2\pi}$ (the CF is the *Fourier transform* of a probability density).

Q2. (i) If $F(t) := \int_0^{\infty} e^{-x} \cos xtdx$,

$$\begin{aligned} F(t) &= \int_0^{\infty} e^{-x} \cos xtdx = - \int_0^{\infty} \cos xt de^{-x} \\ &= -[\cos xt \cdot e^{-x}]_0^{\infty} + \int_0^{\infty} e^{-x} (-t \sin xt) dx \\ &= 1 = t \int_0^{\infty} \sin xt de^{-x} \\ &= 1 + t[\sin xt \cdot e^{-x}]_0^{\infty} - t \int_0^{\infty} e^{-x} \cdot t \cos xtdx \\ &= 1 - t^2 \int_0^{\infty} e^{-x} \cos xtdx = 1 - t^2 F(t) : \end{aligned}$$

$$F(t)(1+t^2) = 1, \quad F(t) = 1/(1+t^2).$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} e^{ixt} \cdot \frac{1}{2} e^{-|x|} dx &= \int_{-\infty}^{\infty} \cos xt \cdot \frac{1}{2} e^{-|x|} dx + i \int_{-\infty}^{\infty} \sin xt \cdot \frac{1}{2} e^{-|x|} dx \\ &= \int_{-\infty}^{\infty} \cos xt \cdot \frac{1}{2} e^{-|x|} dx = 1/(1+t^2), \end{aligned}$$

by above (the second integral is zero: *odd* integrand, symmetric limits. The first integral is twice \int_0^∞ : *even* integrand, symmetric limits.

Thus the characteristic function of the *symmetric exponential* probability density $\frac{1}{2}e^{-|x|}$ is $1/(1+t^2)$.

(ii). Take $\epsilon > 0$. $f(z) = 1/(\pi(1+z^2))$ (to use Jordan's Lemma for $e^{itz}/(\pi(1+z^2))$). The only singularity inside γ is at $y = i$, a simple pole.

$$\text{Res}_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi \cdot 2i} = \frac{-ie^{-t}}{2\pi}.$$

By Cauchy's Residue Theorem:

$$\int_{\gamma} f = 2\pi i \cdot \left(\frac{-ie^{-t}}{2\pi} \right) = e^{-t}.$$

But

$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f \rightarrow \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad (\text{Jordan's Lemma}).$$

This gives the result for $t > 0$. For $t = 0$, it is an arctan (or \tan^{-1}) integral. For $t < 0$: replace t by $-t$. //

Thus the CF of the symmetric Cauchy density $1/(\pi(1+x^2))$ is $e^{-|t|}$.

Q3. The similarity between (i) and (ii) of Q2 is an instance of the *Fourier Integral Theorem*: under suitable conditions, doing the Fourier transform twice gets back to where we started, apart from (a) e^{ixt} first time, but e^{-ixt} the second time; (b) a factor $1/2\pi$. (There are various formulations, depending on the class of function and type of integral – see a good book on Analysis, or a book on Fourier Analysis.) In Q1, the function $e^{-x^2/2}$ is its own Fourier transform (to within the constant factor $1/\sqrt{2\pi}$).

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