spsoln6.tex

Solutions 6. 25.11.2010

Q1. Recall our method: one $U(0,1)$ gives a single sequence of independent coin-tosses by dyadic expansion. Rearranging by the Cantor diagonal argument gives infinitely many sequences of coin-tosses , each of which gives a $U(0,1)$, hence a $N(0,1)$, so we get a sequence of $N(0,1)$s, hence a BM.

Repeating the Cantor diagonal argument: each coin-tossing sequence splits into infinitely many. Then proceeding as above, we get infinitely many [independent] BMs.

Q2. As integration is order-preserving, so is conditional expectation. So

$$E[X_n|\mathcal{B}] \leq E[X_{n+1}|\mathcal{B}] \leq E[X|\mathcal{B}].$$

So $E[X_n|\mathcal{B}]$ is increasing, and bounded above, so has a limit; as $E[X_n|\mathcal{B}]$ is $\mathcal{B}$-measurable, so is the limit. We need to show that this limit is $E[X|\mathcal{B}]$. Now for any $B \in \mathcal{B}$,

$$\begin{aligned}
\int_B \lim E[X_n|\mathcal{B}]dP &= \lim \int_B E[X_n|\mathcal{B}]dP \qquad \text{(Monotone Convergence)} \\
&= \lim \int_B X_n dP \qquad \text{(definition of conditional expectation)} \\
&= \int_B X dP \qquad \text{(Monotone Convergence)} \\
&= \int_B E[X|\mathcal{B}]dP \qquad \text{(definition of conditional expectation).}
\end{aligned}$$

As this holds for each $B \in \mathcal{B}$, $\lim E[X_n|\mathcal{B}] = E[X|\mathcal{B}]$ follows.

Q3. Choose any $B \in \mathcal{B}$. By Fatou's Lemma applied to $X_n.I_B$,

$$\int_B \liminf X_n dP = \int \liminf I_B.X_n dP \leq \liminf \int I_B.X_n dP = \liminf \int_B X_n dP.$$

The extreme left and extreme right here and the definition of conditional expectation give

$$\int_B E[\liminf X_n|\mathcal{B}]dP \leq \liminf \int_B E[X_n|\mathcal{B}]dP.$$

As this holds for each $B \in \mathcal{B}$,

$$E[\liminf X_n|\mathcal{B}] \leq \liminf E[X_n|\mathcal{B}].$$

Q4. Choose $B \in \mathcal{B}$. By dominated convergence applied to $X_n I_B$,

$$\int_B X_n dP \to \int_B X dP.$$

By definition of conditional expectation, this says

$$\int_B E[X_n|\mathcal{B}]dP \to \int_B E[X|\mathcal{B}]dP.$$

As this holds for all $B \in \mathcal{B}$,

$$E[X_n|\mathcal{B}] \to E[X|\mathcal{B}].$$

Note. Just as there are conditional versions of the three convergence theorems, there are also conditional versions of the inequalities (Jensen, Hölder, Minkowski).

Q5. The CF of the symmetric Cauchy is $e^{-|t|}$. So $(X_1 + \ldots + X_n)/n$ has CF

$$E \exp\{(X_1 + \ldots + X_n).t/n\} = \Pi_1^n E \exp\{(X_i).t/n\} = [e^{-|t|/n}]^n = e^{-|t|}.$$

So $(X_1 + \ldots + X_n)/n$ is symmetric Cauchy, as required.

This complete failure of $(X_1 + \ldots + X_n)/n$ to converge to the mean μ as $n \to \infty$, as in SLLN, does not contradict the SLLN as here the mean does not exist: the density is $f(x) = 1/(\pi(1 + x^2))$, so $xf(x) \notin L_1$, so the mean does not exist.

NHB