spsoln7.tex

Solutions 7. 2.12.2011

Q1. For $x, y > 0$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q} \qquad (*)$$

(i) from Solns 3 Q1(ii) with $x_i \to x_i^{\lambda_i}$, then $x_1, x_2, \lambda_1, \lambda_2 \to x, y, 1/p, 1/q$, or (ii) by finding the maximum of the convex function $\phi(x) := xy - x^p/p$ (y fixed): $\phi'(x) = 0$ where $y - x^{p-1} - 0$, $x = y^{1/(p-1)}$. There,

$$\phi(x) = y^{\frac{1}{p-1}+1} - \frac{y^{\frac{p}{p-1}}}{p} = y^{\frac{p}{p-1}}(1 - \frac{1}{p}) = y^{\frac{p}{p-1}}/\frac{p}{p-1} = y^q/q,$$

a maximum (check).

By assumption, $f^p \in L^1$, $g^q \in L^1$, so $fg \in L^1$ by $(*)$ $(x, y \to f, g)$. So all three conditional expectations are defined. By $(*)$,

$$\frac{|f|}{(E[|f|^p|\mathcal{B}])^{1/p}} . \frac{|g|}{(E[|g|^q|\mathcal{B}])^{1/q}} \leq \frac{|f|^p}{pE[|f|^p|\mathcal{B}]} + \frac{|g|^q}{qE[|f|^p|\mathcal{B}]}$$

on the set B where both denominators on LHS are positive. This set B is in $\mathcal{B}$, so we can take $E[.|\mathcal{B}]$ above, to get

$$\frac{E[|fg||\mathcal{B}]}{(E[|f|^p|\mathcal{B}])^{1/p} . (E[|g|^q|\mathcal{B}])^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

on B. As $|E[fg|\mathcal{B}]| \leq E[|fg||\mathcal{B}]$, this gives the result on B.

The set $B_1 := \{E[|f|^p|\mathcal{B}] = 0\} \in \mathcal{B}$. So by definition of conditional expectation,

$$\int_{B_1} |f|^p dP = \int_{B_1} E[|f|^p|\mathcal{B}] dP = 0,$$

so $f = 0$ a.s. on B_1; similarly, $g = 0$ a.s. on the corresponding set B_2 involving g. But since $B_1 \cup B_2 = B^c$, the result holds (and says $0 = 0$) on B^c. So the result holds on $\Omega = B \cup B^c$. //

Q2. The proof in Solns 3 Q2 applies with f, g replaced by $E[f|\mathcal{B}]$, $E[g|\mathcal{B}]$, and Hölder's inequality by the conditional Hölder inequality.

Q3. Recall 'curve above chord' for convex functions. Letting the end-points of the chord approach each other (at x_0, say) gives 'curve above support line' [a tangent where this exists, but the right and left derivatives may differ on some null set]:

$$\phi(x) \geq \phi(x_0) + (x - x_0).A(x_0),$$

where $A(x_0)$ is the slope of the support line at x_0. So

$$\phi(X) \geq \phi(E[X|\mathcal{B}]) + (X - E[X|\mathcal{B}])A(E[X|\mathcal{B}]). \qquad (*)$$

If $E[X|\mathcal{B}]$ is bounded, all three terms are integrable (a convex function ϕ is bounded on compact sets, and similarly for A). Take $E[.|\mathcal{B}]$ of $(*)$: the A-term is $\mathcal{B}$-measurable, so can be taken outside, and then its coefficient vanishes, leaving

$$E[\phi(X)|\mathcal{B}] \geq \phi(E[X|\mathcal{B}]),$$

as required.

In the general case, let $G_n := \{|E[X|\mathcal{B}]| \leq n\}$ ($\mathcal{B}$-measurable). Then $E[I_{G_n}X|\mathcal{B}] = I_{G_n}E[X|\mathcal{B}]$ ('taking out what is known'), which is bounded. So by the bounded case above,

$$\phi(I_{G_n}E[X|\mathcal{B}]) \leq E[\phi(I_{G_n}X|\mathcal{B})]. \qquad (**)$$

The RHS in $(**)$ is

$$E[\phi(I_{G_n}X|\mathcal{B}) = E[I_{G_n}\phi(X) + I_{G_n^c}\phi(0)|\mathcal{B}]$$

(in G_n, both sides are $E[\phi(X)|\mathcal{B}]$; outside it, both are $E[\phi(0)|\mathcal{B}]$)

$$= I_{G_n}E[\phi(X)|\mathcal{B}] + I_{G_n^c}\phi(0)$$

(taking out what is known)

$$\to E[\phi(X)|\mathcal{B}] \qquad (n \to \infty).$$

The LHS of $(**) \to \phi(E[X|\mathcal{B}])$ (convex functions are continuous – we are given this). This extends the result to the general case. //

Q4. $|\int_B f_n - \int_B f| = |\int_B (f_n - f| \leq \int_B |f_n - f|$. Taking sups over B proves the inequality. Next, with $a \wedge b := \min(a, b)$, $|f_n - f| = f_n + f - 2f_n \wedge f$ (check). Integrate: $\int f_n = 1$, $\int f = 1$ as these are densities. As $0 \leq f_n \wedge f \leq f$, integrable, dominated convergence gives $\int f_n \wedge f \to \int f = 1$. So the integral of RHS $\to$ 1+1-2 = 0. So the integral of LHS $\to 0$ also. //

NHB