spsoln8.tex

Solutions 8. 9.12.2011

Q1. (i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a mg. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submg.
(ii) If M is a submg, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\phi(M_s) \leq \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s]$$

(the second inequality by conditional Jensen as above), and again $\phi(M)$ is a submg.

Q2. As BM is a mg and x^2 is convex, Q1 (i) gives B^2 a submg. As $B_t^2 - t$ is a mg [L23],

$$B_t^2 = [B_t^2 - t] + t \qquad \text{(submg = mg + incr)}$$

is the Doob-Meyer decomposition of B_t^2, with increasing process t [the QV].
(ii) For $p \geq 1$, $|x|^p$ is convex (for non-zero x, 2nd derivative $p(p-1)|x|^{p-2} \geq 0$).
(iii) $x^+ := \max(x, 0)$ is convex.

Q3. *Proof (Doob's Submartingale Inequality).* Let

$$F := \{\max_{k \leq n} X_k \geq c\}, \quad F_k := \{X_0 < c\} \cap \{X_1 < c\} \cap \ldots \{X_{k-1} < c\} \cap \{X_k \geq c\}.$$

Then F is the disjoint union $F = F_0 \cup \ldots \cup F_n$. Also $F_k \in \mathcal{F}_k$, and $X_k \geq c$ on F_k. So

$$E[X_n I(F_k)] \geq E[X_k I(F_k)] \quad (X \text{ a submg}) \quad \geq cE[I(F_k)] = P(F_k).$$

Sum over k:

$$E[X_n] \geq E[X_n I(F)] = \sum_k E[X_n I(F_k)] \geq \sum_k cP(F_k) = cP(F).$$

Q4 *Doob's Submg Convergence Th.* For X L_1-bounded, by K say, letting $n \to \infty$ gives $P(X^* \geq c) = P(\sup_n X_n \geq c) \leq K/c \to 0$, so $X^* < \infty$ a.s., which shortens the proof in lectures.

Q5 *(Second Borel-Cantelli Lemma for Pairwise Independence).* For A_n pairwise independent, $\sum P(A_n)$ diverges implies $P(\limsup A_n) = P(A_n \; i.o.) = 1$.

Proof. For A_n pairwise independent, put $S_n := \sum_1^n I(A_i)$, $S := \sum_1^\infty I(A_i)$, $m_n := E[S_n] = \sum_1^n P(A_i)$.

$$var(S_n) = E[(S_n - m_n)^2] = E[(\sum_{i=1}^n (I(A_i) - EI(A_i))(\sum_{j=1}^n (I(A_j) - EI(A_j))] = E[\sum_i \sum_j (\ldots)(\ldots)]$$

$$= \sum_i E[(\ldots)^2] + \sum_{i \neq j} E(\ldots)(\ldots)] = \sum_i E[(\ldots)^2]$$

(the sum over $i \neq j$ is 0, as there by pairwise independence and the Multiplication Theorem $E[(\ldots)(\ldots)] = E[(\ldots)]E[(\ldots)] = 0.0 = 0$ – variance of sum = sum of variances under pairwise independence). As $I(A_i)$ is Bernoulli with parameter $P(A_i)$, its variance is $P(A_i)[1 - P(A_i)] \leq P(A_i)$. So

$$var(S_n) = E[(S_n - m_n)^2] \leq \sum_1^n P(A_i) = m_n,$$

which increases to $+\infty$ as $\sum P(A_n)$ diverges, by assumption. But

$$\begin{aligned} P(S \leq m_n/2) &\leq P(S_n \leq m_n/2) \qquad (S_n \leq S) \\ &= P(S_n - m_n \leq -m_n/2) \\ &\leq P(|S_n - m_n| \geq m_n/2) \\ &\leq \frac{4}{m_n^2} var(S_n) \qquad \text{(by Tchebycheff's Inequality)} \\ &\leq 4/m_n \quad \text{(by above)} \quad \to 0 \quad (n \to \infty). \end{aligned}$$

But the LHS increases to $P(S < \infty)$, by continuity (= σ-additivity) of $P(.)$. So $P(S < \infty) = 0$: $P(\sum I(A_n) < \infty) = 0$, i.e. $P(\sum I(A_n) = \infty) = 1$. This says that $P(A_n \; i.o.) = 1$: $P(\limsup A_n) = 1$. //

Q6 *(Etemadi's SLLN under pairwise independence).* Both places in lectures which assumed independence ('variances add over independent summands') and the second Borel-Cantelli lemma) extend to pairwise independence.

NHB