

Part 1

I. BACKGROUND

1. Revision of the Black-Scholes formula and PDE

We recall the Black-Scholes model, with one risky asset: we have a *bank account* $B = (B_t)$, and a *risky asset* $S = (S_t)$, with dynamics

$$dB_t = rB_t dt$$

(so r is the riskless interest rate, or *spot rate*, or *short rate*; see Ch. III below),

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad (GBM)$$

(‘W for Wiener’, as we are using ‘B for bank’, rather than ‘B for Brownian’), the stochastic differential equation (SDE) for geometric Brownian motion (GBM). Here μ is the mean return rate on the stock, σ is the volatility of the stock, $W = (W_t)$ is the driving noise – Brownian motion (BM), representing the uncertainty or unpredictability in this uncertain and unpredictable world. So our holding at time t is described by a bivariate stochastic process (B_t, S_t) , on a filtered probability space (or stochastic basis) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. A *trading strategy* is a pair of stochastic processes $\phi = (\phi^B, \phi^S)$; here ϕ_t^B, ϕ_t^S are the amounts of cash and stock held at time t . Both processes are *predictable*: the value at time t will be known immediately before t . Left-continuity suffices for this, and this always holds in the Black-Scholes model, where everything is continuous. The *value process* is the process V obtained by following strategy ϕ :

$$V_t(\phi) = \phi_t^B B_t + \phi_t^S S_t; \quad (V)$$

the first term is the cash part, the second the risky-asset part. The *gain process* is

$$G_t(\phi) := \int_0^t \phi_u^B dB_u + \int_0^t \phi_u^S dS_u, \quad (G)$$

the net gain (profit or loss) from following strategy ϕ . The strategy is *self-financing* (SF) if the change in G is due only to changes in B and S (so that

the trader can trade with no need for extra funds from his firm, and no profit diverted for his/his firm's use):

$$dV_t(\phi) = dG_t(\phi),$$

i.e.

$$dV_t = d(\phi_t^B B_t + \phi_t^S S_t) = \phi_t^B dB_t + \phi_t^S dS_t. \quad (SF)$$

A *contingent claim* Y is just an $\mathcal{F}_T$ -measurable random variable (here T is the *expiry time*; think of Y as the payoff (to the holder) or claim (to the writer) of an option expiring at time T ; this is contingent (= dependent) on what happens – whether the option expires in/at/out of the money (ITM/ATM/OTM), etc. A (contingent) claim is *attainable* if there exists a SF strategy ϕ attaining it, i.e. such that

$$V_T(\phi) = Y.$$

Then ϕ *generates* Y , and $V_t(\phi)$ is the *price* of Y at time t .

Derivation of the Black-Scholes PDE

For an attainable claim on the stock S , its value V_t at time t depends in the stock price S_t ,

$$V_t = V(t, S_t).$$

For V suitably smooth – $V \in C^{1,2}([0, t] \times \mathbb{R}^+)$ – Itô's lemma gives

$$dV(t, S_t) = \left(\frac{\partial V}{\partial t}(t, S_t) + \frac{\partial V}{\partial S}(t, S_t)\mu S_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S}(t, S_t)\sigma S_t dW_t \quad (Ito)$$

(as in MATL480 Ch. V: this uses (GBM) and $dW_t^2 = dt$). For $t \in [0, T]$, define

$$\phi_t^S := \frac{\partial V}{\partial S}(t, S_t), \quad \phi_t^B := (V_t - \phi_t^S S_t)/B_t;$$

then (V) holds, so the value of this strategy is V . If ϕ is SF (and we will only need to consider SF strategies), (SF) gives, substituting the above,

$$\begin{aligned} dV_t &= \phi_t^B dB_t + \phi_t^S dS_t \\ &= (V(t, S) - \frac{\partial V}{\partial S}(t, S_t))rdt + \frac{\partial V}{\partial S}(t, S_t)S_t\mu dt + \sigma dW_t. \end{aligned}$$

This and (SF) give us two expressions for dV_t . The coefficients of dW_t are the same; equating the coefficients of dt gives

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2 = rV - r\frac{\partial V}{\partial S} + \frac{\partial V}{\partial S}\mu S_t.$$

This gives the famous *Black-Scholes PDE*:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} = rV. \quad (BS - PDE)$$

We solved this in MATL480 VI.3, for the two cases of payoff (terminal condition at $t = T$)

$$V(T, S) = (S - K)_+ \quad (BC)$$

(European call option with strike K), giving

$$V(t, S) = S\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-), \quad d_{\pm} := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)] / \sigma\sqrt{T-t}, \quad (C)$$

and the corresponding formula (P) for European put options with strike K , where instead

$$V(T, S) = (K - S)_+.$$

Again as in MATL480 VI, the Feynman-Kac theorem allows one to express the solution of the PDE + BC for $V = V(t, x)$,

$$\frac{\partial V}{\partial t} + b(x)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma(x)^2\frac{\partial^2 V}{\partial x^2} = rV, \quad V(T, x) = f(x) \quad (PDE)$$

as

$$V(t, x) = e^{-r(T-t)}E_{t,x}^{\mathbb{Q}}[f(X_T)|\mathcal{F}_t], \quad (RNVF)$$

where under the probability measure $\mathbb{Q}$ the diffusion process X has dynamics, starting from $X = x$ at time t ,

$$dX_s = b(X_s)ds + \sigma(X_s)dW_s^{\mathbb{Q}}, \quad s \geq t, \quad X_t = x$$

(here $W^{\mathbb{Q}}$ is standard BM under $\mathbb{Q}$). Specialising to

$$b(x) = rx, \quad \sigma(x) = \sigma x$$

(so the general PDE becomes the BS PDE), this gives: the unique no-arbitrage price of the claim $Y = (S_T - K)_+$ (European call option) at time $t \in [0, T]$ is

$$V_{BS}(t) = E^{\mathbb{Q}}[e^{-r(T-t)}Y|\mathcal{F}_t];$$

here $\mathbb{Q}$ is the *equivalent martingale measure* (EMM) – the probability measure $\mathbb{Q} \sim \mathbb{P}$ under which the risky-asset price $S_t/B_t = e^{-rt}S_t$ has $\mathbb{Q}$ -dynamics

$$dS_t = S_t[r dt + \sigma dW_t^{\mathbb{Q}}].$$

Girsanov's theorem

As we covered this in MATL480, we can be informal here. On a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, consider an SDE (under the measure $\mathbb{P}$)

$$dX_t = b(X_t)dt + v(X_t)dW_t, \quad X_0 = x_0, \quad \mathbb{P}.$$

Under the relevant technical conditions, define a measure $\mathbb{Q}$ by its Radon-Nikodym derivative w.r.t. $\mathbb{P}$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp\left\{-\frac{1}{2} \int_0^t \left(\frac{b^{\mathbb{Q}}(X_s) - b(X_s)}{v(X_s)}\right)^2 ds + \int_0^t \left(\frac{b^{\mathbb{Q}}(X_s) - b(X_s)}{v(X_s)}\right) dW_s\right\}.$$

Then under $\mathbb{Q} \sim \mathbb{P}$,

$$dW_t^{\mathbb{Q}} = -\left(\frac{b^{\mathbb{Q}}(X_t) - b(X_t)}{v(X_t)}\right)dt + dW_t$$

is a BM, and on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ X satisfies the SDE

$$dX_t = b^{\mathbb{Q}}(X_t)dt + v(X_t)dW_t^{\mathbb{Q}}, \quad X_0 = x_0, \quad \mathbb{Q}.$$

Note that it is only the *drift* that changes (b becomes $b^{\mathbb{Q}}$). The *diffusion coefficient* v is the same. Indeed, if the diffusion coefficients were different, the two probability measures would not be equivalent. Statistically, what this means is that, given two diffusions with the same diffusion coefficient, we can test for whether their drifts are the same by using a likelihood-ratio test, the test statistic being obtained from the RN derivative above.

So: if we use Girsanov's theorem to move from

$$dS_t = S_t(\mu dt + \sigma dW_t) \quad (GBM - \mathbb{P})$$

to

$$dS_t = S_t(r dt + \sigma dW_t), \quad (GBM - \mathbb{Q})$$

the RN derivative is

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{-\frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T - \left(\frac{\mu - r}{\sigma}\right) W_T\right\}.$$

Recall also from MATL480:

No-Arbitrage Theorem. The market has no arbitrage (is NA) iff EMMs *exist*.

Completeness Theorem. An NA market is complete (i.e. all contingent claims can be replicated) iff EMMs are *unique*.

In reality, **markets are never complete**.

Roughly speaking a market is complete if there are as many assets as independent sources of randomness. But: while there are lots of assets, the uncertain world is so complicated that it contains even more sources of randomness. So: we have to live with incompleteness, non-unique prices (just like out on the High Street!), and bid-ask spreads.

The SF strategy ϕ enables the option seller to find a *perfect hedge*, which will cover him against any claim the option buyer may make. This is called *delta-hedging*. For, recall ‘the Greeks’, the most important and basic of which is Delta, $\Delta := \partial V / \partial S$ (below). However, the option seller only sells the option in the hope of making money by doing so. If he hedges perfectly, he will make no loss – but he will *make no profit* either. So he might as well not bother selling the option in the first place. In practice, he will normally use *partial hedging* – lay off some of the risk, but not all, so as not to lose all potential profit. ‘Nothing venture, nothing win’!

Three Greeks. Recall:

Delta,

$$\Delta := \partial V / \partial S;$$

Gamma,

$$\Gamma := \partial \Delta / \partial S = \partial^2 V / \partial S^2;$$

Theta, which measures sensitivity to time:

$$\Theta_t := \partial V_t / \partial t = -\partial V_t / \partial (T - t)$$

(in terms of the remaining time $\tau := T - t$ to expiry). These are linked. For, recall (*Ito*) (Itô in text, but ‘Ito in TeX’):

$$dV(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2}(t, S_t)\sigma^2 S_t^2 dt.$$

One can see from this equation that it will imply a link between Delta, Gamma and Theta.

To derive this, recall $dS = S(\mu dt + \sigma dW)$, so $(dS)^2 = \sigma^2 S^2 (dW)^2 = \sigma^2 S^2 dt$. This can be written

$$dV = \Theta dt + \frac{1}{2} \Gamma \sigma^2 S^2 dt + \Delta dS.$$

But from the self-financing condition (SF), we also have (where $B = B_t$ is the bank account, so $dB_t = r_t B_t dt$)

$$dV = [(V - \Delta S)/B] dB = (V - \Delta S) r dt + \Delta dS.$$

Equating, the three Greeks are linked to the spot rate by ($dB_t = r_t B_t dt$)

$$r_t V(t, S_t) = r_t \Delta_t S_t + \Theta_t + \frac{1}{2} \Gamma_t \sigma^2 S_t^2.$$

Note 1. End-of-year assessment of 2017: A stock-market high plus a zombie economy?

To understand the world, you need to follow the news, and commentary on it. End-of-year assessments are a good start. See, e.g.:

“*Guardian*, 30.12.2017, Financial (p.23): “Stock markets celebrate annual gains of \$9tn dollars as 2017 ends with record highs around globe.”

This is odd, as the world’s economies have still not recovered from the Crash of 2007/8/...

Analysis: Five reasons for the markets’ bumper year.

Boom in global trade. Almost a decade since the financial crisis, 2017 was the year global growth came back with a bang. The failure by rightwing populists to seize power in Europe led to expectations of political stability, helping the single-currency bloc (euro) to recover. China kept up its rate of expansion despite fears over a slow-down. According to the OECD, global real trade growth accelerated from 2.6 % to 4.8 % this year and GDP growth jumped from 3.1 % to 3.6 %.

Loose tax and monetary policy. Low interest rates and tax cuts stimulated demand for shares. Donald Trump’s corporate tax cuts are expected

to boost US company profits. Central banks kept pumping money into the financial system through quantitative easing. The debt-buying programmes have caused a fall in bond yields, forcing market professionals to hunt for greater returns from riskier assets – with stocks the investment of choice.

Low volatility. Wall Street’s “fear index”, the Chicago Board Options Exchange volatility index (VIX), has fallen to record lows. It measures investor expectations for price swings in the stock market over a 30-day period. It fell below 9 points in July and has not gone anywhere near the 20 that points to falling markets.

Weak pound. The FTSE 100, full of companies earning much of their profit in foreign currencies, has surged thanks to stronger global growth and a weak pound – almost 10% down on the dollar since the Brexit vote.

Irrational complacency. A general sense of confidence among investors over the state of the global economy – and of geopolitics – has been significant. However, they may be ignoring problems just beneath the surface. The fund manager Alberto Gallo at Algebris Investments thinks there are risks arising from geopolitics, central bank policy and inflation. Markets barely blinked this year, despite faltering Brexit talks and concerns over North Korea’s weapons. Given the rising number of one-sided bets on the market to keep rising, the risks of greater volatility could be growing.

Richard Partington.”

The *second* of these five points is crucial for us (QE, etc.: II.2, W2a). But note the preponderance of such things as geopolitics and voter and investor psychology – both difficult if not impossible to model mathematically. This illustrates why Economics is as much (or more) an art than a science.

Postscript. One should cap the end-of-year assessment of 2017 (retrospective) with the beginning-of-year assessment of 2018 (prospective). See Problems 1, and e.g.

Nils Pratley: Shares have gone through the roof: could they possibly go even higher? As 2018 gets under way, there are clear signs that stocks are overvalued. So is it time to sell? Nils Pratley weighs up the situation and talks to two respected fund managers. *Observer*, Sunday 7.1.2018, Business, p.38-39.

Note 2. A (planned) replacement for LIBOR: SONIA

Recall (MATL480, I.1 W1a; see also I.4 W1b below) the London Inter-Bank Offer Rate (LIBOR). Bank Rate is the rate at which the Bank of England (BoE) lends to banks. It is changed fairly rarely, by BoE in consultation with the Treasury as representing the Government (HMG). As is well known, it has been at historically low levels ($< 1\%$) since the Crash ten years ago; it will be treated as riskless here. LIBOR is one of the most widely used measures of the rate at which banks lend to each other. It changes daily, is higher than Bank Rate, and is available over a number of time-periods (terms). Although there is no such thing as an absolutely riskless rate of interest, nevertheless LIBOR serves as a benchmark for one (the r in the Spot-Rate Models of Ch. III; cf. the LIBOR Market Models of Ch. V.3).

However, as we saw in MATL480, there was a major and very damaging scandal involving rigging of LIBOR (illicit market manipulation – ‘collusion pretending to be competition’), inevitably known as the “Lie-bor” scandal. Hardly surprisingly, it is now planned to replace LIBOR, by something less vulnerable to such market manipulation – SONIA (Sterling Overnight Index Average). See the pieces below:

Ben MOSHINSKY, *Business Insider*, 16.10.2017:

LONDON The Bank of England will start setting the interest rate benchmark to replace Libor in April 2018.

The central bank will take over the administration of the SONIA rate – which stands for the Sterling Overnight Index Average – on April 23.

A panel of banks including Goldman Sachs, Barclays and Deutsche Bank voted to recommend an alternative to Libor for use in sterling derivatives in April of this year.

A Bank of England working group approved SONIA as its preferred short-term interest rate benchmark thereafter.

The SONIA index tracks the rates of actual overnight funding deals on the wholesale money markets, rather than relying on submitters like the Libor benchmark does. SONIA’s use will minimise “opportunities for misconduct,” the Bank of England said at the time of the working group’s decision.

The group chose the SONIA rate with a more than two-thirds majority “based on robust transaction volumes” and how well it measures overnight interest rates “that are considered close to risk-free”. SONIA has been around for 20 years.

The Libor manipulation scandal, which hit four years after the 2008 financial crisis, risked breaking trust in London's ability to function as a financial centre.

Libor, a benchmark underpinning more than \$ 300 trillion in loans and derivatives, was set by a panel of banks that submitted short-term borrowing rates to the British Bankers' Association at the time of the scandal.

Regulators found that traders and rate-setters had colluded to shift the rate, benefiting the traders' positions and leading to billions of pounds in fines from US and UK regulators. Libor has since been reformed and is no longer compiled by the BBA.

The Financial Stability Board, which is chaired by BoE Governor Mark Carney, started to reform interest rate benchmarks in 2014 and the central bank set up its working group on Libor alternatives in 2015.

29.11.2017: LONDON (Reuters), Reporting by David Milliken and Huw Jones; Editing by William Schomberg and David Evans.

The Bank of England has stepped up efforts to replace the scandal-hit LIBOR interest rate benchmark with the SONIA measure of overnight rates as the main benchmark for commercial sterling interest rates by the end of 2021. "It has become increasingly clear that we cannot rely on LIBOR in the long term", BoE Governor Mark Carney said in a speech to bankers in London on Wednesday.

British regulators have already said they want a switch to SONIA by the end of 2021, and on Wednesday the BoE announced changes to ensure transition stays on track in the face of some market reluctance.

Carney said there was a risk that the banks which currently quote interbank lending rates that are used to calculate LIBOR might pull out and precipitate the benchmarks collapse, which he added "raises obvious financial stability concerns".

The Financial Conduct Authority announced a deal this month whereby banks have committed to submitting quotes for compiling LIBOR until the end of 2021.

LIBOR – a measure of interbank lending rates for various time periods – is used as the basis for trillions of pounds worth of contracts. But during the financial crisis, some traders manipulated the rate, leading to criminal convictions and huge fines for international banks.

The BoE said it would broaden a working group which would steer the transition away from LIBOR to include fund managers and non-financial

companies that issue debt, as well as bankers.

Bank of England Deputy Governor for banking and markets Dave Ramsden said the change would “catalyse” transition to SONIA across a broad range of markets.

Barclays (BARC.L) chief compliance officer Francois Jourdain would remain chair of the group, aided by Shell (RDSa.L) finance executive Frances Hinden and Legal & General Investment Managements (LGEN.L) Simon Wilkinson as vice-chairs. “The transition to SONIA is necessary and not optional”, Jourdain said.

There is reluctance in parts of the market to switch to SONIA, with participants saying it is not appropriate for longer term contracts. SONIA is an overnight rate, while Libor covers “terms” or periods going out decades into the future.

A key near-term priority for the working group will be to make recommendations relating to the potential development of term SONIA reference rates. “This work is already under way and a public consultation is planned for the first half of 2018”, the BoE said. Scott OMalia, chief executive of ISDA, the global derivatives industry body, said it was incumbent on the industry to find solutions on transitioning. “Its not an option to fail”, OMalia, a former U.S. derivatives regulator, said.

Note 3. Links between stock markets (MATL480) and bond markets (MATL481)

Market Update: “Stocks and Bond Markets Could Fall Together”. Freddie Lait of Latitude warns that this crash could be different – equities and bonds could lose value at the same time. So make sure you have portfolio diversifiers.

.... Lait: What’s happened is, broadly, even a year ago everyone was saying the markets are very expensive and we were all getting slightly nervous about where they were going to go. And I think when we spoke a year ago, I said, expected returns would be quite low and expected volatility should probably be quite high.

Now, in one sense, we’ve seen that. But in another, we’ve been surprised in both ways that actually the markets carried on grinding upwards double-digits in most regions, which makes it incrementally more expensive, and volatility at an index level [average of stocks] has been very low.

But that said, the opportunities that are coming up for us mostly are because at a stock level [individual stocks] the volatility has been incredibly

high. And so, many of our stocks are moving up or down 30%, 40%. You see lots of great stocks moving down 20% on earnings, etc. So, I think, there are lots of the volatility at the single stock level does give you opportunity in the value sectors in particular.

Wall: And is that why you have, well, quite a sizable holding in cash? You've got 20% in cash at the moment. Is that because just you're sitting in the sidelines and waiting for those opportunities?

Lait: Yes. So, I think, now is the time not to take a large amount of risk. I'm still a big believer that good stocks and good companies will continue to compound wealth for us as shareholders and for my clients for the next 5 to 10 years. But I do think that when you run an unleveraged fund, having the cash tactically to be able to buy and take advantage of the underlying volatility is incredibly important and gives the portfolio a nice bit of velocity if we ever do get a pullback [market downturn].

Wall: And are you expecting that pullback?

Lait: I am in a broad sense, but sort of broadly have been for a year or two as well. I think it wouldn't surprise anybody if the markets fell 10% to 15%. That said, calling the timing of one of those is impossible in my view. And so, we always say we've still got 45% invested in equities. And I believe the market can carry on going at a very low level for a very long time.

Again, I think, it's very key to say we have a short list of stocks that we'd like to buy, and those stocks have been falling and rising more rapidly than the market itself. So, while we may or may not have the underlying index pullback, we see the opportunities the stocks we are looking at.

Wall: And do you think when that pullback comes, it is going to be developed markets? Or are we going to have another situation like we did with the Global Financial Crisis where actually it's across the board [that] valuations do slump?

Lait: So, valuations are very high broadly everywhere. And America went first and then Europe caught up and now the emerging markets this year have had a phenomenal run. I think if you saw more than, say, a 10% pullback, which most people would say would be healthy, you definitely expect things to start to re-correlate and this decoupling idea would prove false again like it did in 2007 and like it has in broadly every other crisis.

Markets are expensive. If bond yields rise or growth stalls or inflation comes back, or political unrest grows or any of the other things we worry about happen, I think all markets could fall together. And critically this time, not just stock markets but bond markets, too.

For the last 15 years, bonds and equities have been very decorrelated and that has caused the sort of balanced portfolio to do phenomenally well for the last 15 years. I think next time that might be a real danger when people lose money on both their growth of their portfolio in their stocks as well as their defensive side in their bonds and people need to be much more careful about portfolio construction.

Wall: And presumably, that's why you hold assets such as gold, which are less correlated with equities and bonds?

Lait: Exactly right. So, the things that we have instead on our sort of more diversified side of the portfolio is invested in gold and then some gold-linked emerging market currencies and other emerging market currencies as well, which we see as benefiting from a weaker dollar, a slowdown in central global growth and also possibly benefiting from that strength in the emerging markets that we've been talking about on catchup basis to the other parts of the world.

Note 4. Bankruptcy

We touched on bankruptcy in MATL480 Ch. II. The public limited company (plc) limits the liability of a shareholder to their stake in the company, if the company fails. Then, creditors will be paid in part but not in full by the liquidators. So in essence, *bankruptcy is a mechanism for writing off debt*. This is why the plc was introduced so late (mid-19th C.), and was opposed. Recall that previously, a debtor could be imprisoned (the Marshalsea in London [Dickens, *Little Dorrit*], the Debtors' Prison in York, etc.), until he paid his debt. This was a complete failure: how can someone earn to pay off a debt when in prison? Its advantages include making investment safer (by removing this threat), so encouraging business, and allowing the economy to grow. Its drawbacks include the *moral hazard*, of reckless (or worse, unscrupulous) financial behaviour. Note that President Trump of the USA, a very rich man, has been bankrupt four times:

'Trump doesn't deny that four of his businesses have filed for bankruptcy. He argues, however, that filing for bankruptcy is a common business decision, and he was smart to make the moves when he did. "Hundreds of companies" have filed for bankruptcy, Trump said earlier in the debate. "I used the law four times and made a tremendous thing. I'm in business. I did a very good job" '.

Draw your own conclusions.