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2. $\mathbb{P}$-measure, $\mathbb{Q}$-measure and pricing kernels

Recall (MATL480, Ch. II) that Radon-Nikodym derivatives obey the same rules as derivatives in ordinary calculus: for $\mathbb{P} \sim \mathbb{Q}$,

$$\frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q} = d\mathbb{P}; \qquad \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} = d\mathbb{Q}.$$

So in the RNVF, where we have $E^{\mathbb{Q}}[.]$, which is $\int_\Omega [.] d\mathbb{Q}$, we can replace this by $\int_\Omega [.].d\mathbb{Q}/d\mathbb{P}.d\mathbb{P}$. So if we write

$$\zeta := \frac{d\mathbb{P}}{d\mathbb{Q}}, \qquad \zeta^{-1} := \frac{d\mathbb{Q}}{d\mathbb{P}},$$

the *pricing kernel*, we can write the RNVF as

$$V(t,x) = e^{-r(T-t)} E_{t,x}[f(X_T)\zeta|\mathcal{F}_t], \qquad (RNVF - \mathbb{P})$$

as a $\mathbb{P}$-expectation, under E, rather than a $\mathbb{Q}$-expectation, under $E^{\mathbb{Q}}$, at the price of introducing an extra factor ζ into the integrand. It is often convenient to do this.

Note. The 'pricing' in the name is evident (risk-neutral *valuation*). For 'kernel': this derives from the subject of *integral equations* (rather like differential equations, but with integrals rather than derivatives), where one typically encounters equations such as

$$\int f(y)k(x,y)dy = g(x),$$

to be solved for the unknown function f with g and k given; here k is called the *kernel*.

Notation.

Because nearly all our expectations here in MATL481 will be under $\mathbb{Q}$, it is convenient to drop the $\mathbb{Q}$ in $E^{\mathbb{Q}}[.]$, $E_t^{\mathbb{Q}}[.]$ and just write $E[.]$, $E_t[.]$. Brownian motion (BM) under $\mathbb{Q}$ will be written $W = (W_t)$.

It is then convenient to recognise the primacy of $\mathbb{Q}$ over $\mathbb{P}$, and replace $\mathbb{P}$ in our notation by $\mathbb{Q}^0$. Then $\mathbb{P}$-expectation and BM will be written

$$E^0[.], \qquad W_0$$

(compare $\mathbb{P}$, $\mathbb{P}^*$ in MATL480). The superscript 0 in E^0 (subscript in W_0) comes from *measures of location* in Statistics, related to choice of *origin* 0 on the line (super not sub for E, as we write $E_t[.]$ for $E[.|\mathcal{F}_t]$ later). This change of origin corresponds to the change of *drift* in Girsanov's theorem.

Terminology.

We call $\mathbb{Q}$ the *risk-neutral* measure, $\mathbb{P} = \mathbb{Q}^0$ the *real-world*, *objective* or *physical* measure. Each is useful, but for different purposes; we return to the interplay between these two aspects in Ch. III.

$\mathbb{P}$, $\mathbb{Q}$ *and crises*

The measure $\mathbb{P}$, the objective or real-world measure, is also called the *historical measure* – it looks backwards. Prediction is irrelevant to this – and that is a serious matter when things are about to go seriously wrong, as in the build-up to and onset of a financial crisis. By contrast, the measure $\mathbb{Q}$, the risk-neutral measure, takes account of the *market*, because it deals with *prices*. Price is determined by *trading*, which involves a willing seller selling to a willing buyer – a highly non-trivial human interaction. Prices, and so $\mathbb{Q}$, are sensitive to *sentiment* – how a market (or, the market) is feeling collectively. This is a matter of psychology, and of confidence, as much as of objective fact. This is hardly surprising: money itself ultimately rests on confidence.

The difference between $\mathbb{P}$ and $\mathbb{Q}$ shows up dramatically at times of crisis. Perhaps the most spectacular crisis involving an individual firm was the collapse of Lehman Brothers on 14.9.2008. The credit spreads there between $\mathbb{P}$ and $\mathbb{Q}$ were dramatically large – and this is typical of what happens in a crisis. For, $\mathbb{P}$, the historical measure, looks backwards. But $\mathbb{Q}$, the risk-neutral measure, reflects *prices*, market *sentiment*, and *confidence* – all of which can change rapidly as a crisis develops!

3. History

History: Interest rates

There is a good account of the history of interest, from antiquity to modern times, in James & Webber [JW, Ch. 2]. As part of their conclusions, these authors, writing in 2000, conclude (p.37) that 'there is no such thing as a risk-free [interest] rate'. This conclusion is confirmed by more recent events, and particularly the Crash of 2007/8 and its aftermath; see below. Nevertheless, we shall study *risk-free interest rates* for much of this course. One has to learn to walk before one learns to run ... .

History: Options

Options go back to antiquity. The first mathematician for whom we have a result named after him was the ancient Greek Thales (theorem of Thales: an angle in a semi-circle is a right-angle). Thales is also the 'father of options': around 580 BC, Thales bought options on the future use of olive presses (for making olive oil). When there was an abundant olive crop (as he had predicted), and presses were in high demand, he made a fortune. (Thales is also considered the father of the sciences and of western philosophy.)

Moving forward to the modern world:
Louis Bachelier (1870 - 1946) was the first to introduce Brownian motion into finance, to study option prices (hence the name Bachelier Society for one of the main societies in mathematical finance).

More recently, subject to the assumptions of an idealized market (no arbitrage, etc.), Fischer Black (1938-1995) and Myron Scholes (1941-) derived their formula of 1973 by showing that the option price satisfied a partial differential equation (PDE), of parabolic type (a variant of the *heat equation*). In 1973 Robert Merton (1944-) gave a more direct approach. Meanwhile, 1973 was also the year when the first exchange for buying and selling options opened, the Chicago Board Options Exchange (CBOE).

The Black-Scholes formula led to an explosive growth in financial derivatives, used nowadays by banks and companies world-wide. This mathematical result has contributed to creating a vast new market: the derivatives market worldwide has reached 708 trillion dollars (US GDP: 15 trillion).

Subsequent events:
1997, Nobel Prize for Scholes and Merton (Black, the genius, had died);
1998, Long Term Capital Management, a US hedge fund, had to be bailed out, with huge losses (Scholes and Merton were both on the board of LTCM at the time);
2007/2008 crisis (ongoing): within a month in 2008: Fannie Mae, Freddie Mac, Lehman Brothers, Washington Mutual, Landbanki, Glitnir and Kaupthing, Merrill Lynch (and in the UK, Northern Rock).

Following all this, the market broke up, and interest rates that used to be very close to each other and were used to model risk-free rates for different maturities started to diverge.

History: The Business Cycle

The traditional view here is that when the economy was expanding – 'boom', with demand and activity increasing – firms would compete for labour, wages would rise, costs would rise, prices would rise, inflation would rise. The central bank – Bank of England (BoE) in UK – would *increase* interest rates – Bank rate – to make borrowing money more expensive. This would decrease the demand for borrowing by business, and the economy would contract. By contrast, when the economy was contracting – 'bust', or 'slump' – the Bank would reduce interest rates, to make it cheaper for business to borrow. This would have the effect of making business expansion cheaper; businesses would tend to expand. The expansion, once under way, would tend to overshoot the natural mean position, leading to the next expansionary phase and the next business cycle. There is a good deal of theory on such business cycles. However, since 2007/08 the economy has been consistently flat. In an effort to promote growth, the authorities have held interest rates at historically low levels for long periods. In the UK, bank rate is now 0.5%, up from 02.5%, unprecedentedly low. The authorities have also resorted to unconventional monetary measures, such as *quantitative easing (QE)*, usually described informally as creating electronic money. This has had the desired effect of moving the economy back towards normal, from the crisis of the Crash and its immediate aftermath. But, QE has had undesirable and unpredicted effects. In particular, it has led to a large increase in asset prices. This had benefited those who hold assets – principally, the already affluent. This has widened the gap between the rich and the poor, decreasing social mobility and increasing social and political tensions. In addition, low interest rates have penalised savers. This is both unfair to them, and undesirable nationally: we suffer from an excess of consumer indebtedness, so saving should be encouraged.

History: The Crash of 2007/08 and after – and before ...

The Crash has changed our view on all sorts of things – economic, financial, political, and indeed *interest rates*, the subject of this course! Some of

my views were published in my paper (available in the journal, and on my homepage, under the link to Papers):
[B] N. H. BINGHAM, The Crash of 2008: A mathematician's view. *Significance* **5** (2008), 173-5, MR2654655.
Some comments on events since follow.

Persistent depression

The major western economies have been very slow to recover from the Crash of 07/08. This is not unprecedented: the Japanese economy has had similar – and worse – experiences. Japan was devastated during World War II. After it, and American occupation, the Japanese economy experienced an 'economic miracle', similar to that in Germany (the *Wirtschaftswunder*). From the late 50s to around 1990, Japan had a dominant position in several areas of manufacturing: ship-building (especially oil tankers and super-tankers), steel, cars, electronics (from transistor radios on), etc. There was then a financial crisis – with hindsight, perhaps a precursor of the western Crisis in 2007/08, which involved an asset-price bubble – bubbles burst! The economy was stagnant throughout the 90s, which were described as Japan's lost decade. But things have been little better later (lost decade*s*). Another major factor has been *globalisation* (VI.7). How to cope with such persistent depressions is the source of ongoing economic and political debate, and controversy (Brexit, Trump, etc.)

Over-reaction, and 'getting stuck'.

Markets typically over-react, once they start to react. This is (at least in part) a reflection of the two things that, as is well known, really get markets moving: *fear* and *greed*. These are both quite natural; neither brings out the best, in people or in institutions.

As for 'getting stuck': it is common for a serious slump to take a very long time to recover from. We have our own post-Crash experience; we have the Japanese experience in the 1990s and 2000s (above). Further back, we have the US experience of the Slump (or Depression), following the Wall Street Crash of Tuesday 29 October 1929. This scarred the American psyche (and our own) so badly that it is still remembered:
books – read John Steinbeck, *The Grapes of Wrath* (if you haven't read it already – if you have, re-read it);

songs – listen to *Buddy, can you spare a dime* (Bing Crosby, 1932), on YouTube), etc.
It led to the election of Franklin D. Roosevelt as US President in 1932; his New Deal helped the US economy to recover partially. But what really kick-started the US economy, and cured the Slump, was World War II, and its massive demands for munitions etc. – for which we have to thank (if that is the word) the Japanese attack on Pearl Harbor (7.12.1941).

The US economy had its Crash in 2007-8, and has not fully recovered. Hence the resentment by those who feel excluded, which led to the 2016 election of Trump as President.

4. Assumptions

Multiple curves
LIBOR

This is the *London Inter-Bank Offer Rate* – the rate at which banks lend to each other (at various maturities). This is set by taking the average of quotes from the participating leading banks, and used to be considered reliable. However, there has been illegal market manipulation (price-fixing – the so-called *Lie-bor scandal*). See e.g., in addition to [VF1] (1a)
[VF2] Liam VAUGHAN and Gavin FRENCH, How bankers fixed the world's most important number. The long read, *The Guardian*, 18 January 2017.
Overnight Indexed Swaps (OIS).

These were introduced in the mid-90s. Maturities range from 1 week to 2 years or longer. They are based on the *overnight rates*, used by banks to lend to each other for a day or two. These are harder to manipulate than LIBOR (some are quoted by central banks), and as the loan period is short there is little credit risk.

Nowadays – still in the aftermath of the Crisis of 2007/8 – it is no longer realistic to ignore credit risk and liquidity effects in interest-rate modelling – in effect, pretending that there *is* a risk-free rate governing the LIBOR and inter-bank markets. OIS is a partial solution, as it is the best proxy for a (non-existent) default- and liquidity-free interest rate. But some credit-risk and liquidity effects remain, and show up especially in stress-testing under strong stress scenarios.
SONIA.

See I.1, W1a, Note 2.

We will return to these problems – credit-risk and liquidity effects – later (VI). But next, we need to consider the classical theory.

We can think of multiple curves in interest-rate theory as analogous to multiple prices – bid-ask spread – in incomplete markets. Although real markets are incomplete, so real prices are not unique, we deal with complete markets first (we learn to walk before we learn to run!), as we did in MATL480.

So until further notice, we assume:
no credit risk; no liquidity risk; no multiple curves.

5. Are interest rates positive?

There are two good reasons *not* to lend money:
(a) it deprives one of the use of one's own money (for the duration of the loan);
(b) there is the possibility that one may not get it back – of default (see Ch. VI).
Of course, it is common to lend to a family member, close friend etc. – human beings are social animals. But to lend to an unknown stranger, in a business environment, is quite a different matter. One will need some *inducement* to do so – and this is where *interest*, and *interest rates*, come in.

In an extended slump, business activity is flat, and dangerous. So it is dangerous for banks to lend to business (which is what is needed to kick-start things), because of the risk of default (VI). It is much safer for banks to lend to the central bank (Bank of England here), effectively, to the Government – as this will not default. So it is perfectly reasonable for the central bank to *charge banks for looking after their money* – to punish them for not lending it to business, or to induce them to do so.

Inflation and deflation

With Bank Rate so low – at 0.5 %, after years at 0.25 %, ridiculously low by historical standards – and in view of frictional costs, real interest rates are effectively negative. This is potentially very dangerous, because of the risk of *deflation*: if prices are falling, people may defer buying till later, to get things cheaper; the economy will then freeze up even worse, exacerbating the whole problem ... . Negative interest rates have actually happened ...

The damaging effects of *inflation* are well-known (e.g., the hyper-inflation in Germany and Austria post-WWI devastated their economies, and so their societies and political systems, and paved the way to the rise of Nazism, so to WWII). Governments and central banks need to steer a middle course between these two! – *media via tutissima* (the middle of the road is the safest, Latin).

6. Econometrics; macroeconomic policy

Economic data typically arrive at regular time-intervals – monthly, quarterly or annually. Statistical data of this kind, where *time* is crucially relevant, are known as *time series (TS)*, widely studied in Statistics (see e.g. the SMF (Statistical Methods for Finance) link on my homepage, Ch. V). They are the principal data sources used by government (Treasury) and central banks (BoE) to determine macroeconomic policy – e.g., to steer a middle course between the opposing dangers of inflation and deflation (above). One of the key concepts here is *cointegration*. Cointegrated series are series that move together, and commonly occur in economics. These concepts arose in econometrics, in the work of R. F. ENGLE (1942-) and C. W. J. (Sir Clive) GRANGER (1934-2009) in 1987. Engle and Granger gave (in 1991) an illustrative example – the price of tomatoes in North Carolina and South Carolina. These states are close enough for a significant price differential between the two to encourage sellers to transfer tomatoes to the state with currently higher prices to cash in; this movement would increase supply there and reduce it in the other state, so supply and demand would move the prices towards each other.

Engle and Granger received the Nobel Prize in Economics in 2003. The citation included the following: "Most macroecomomic time series follow a stochastic trend, so that a temporary disturbance in, say, GDP has a long-lasting effect. These time-series are called non-stationary; they differ from stationary series which do not grow over time, but fluctuate around a given value. Clive Granger demonstrated that the statistical methods used for stationary time series could yield wholly misleading results when applied to the analysis of nonstationary data. His significant discovery was that specific combinations of nonstationary time series may exhibit stationarity, thereby allowing for correct statistical inference. Granger called this phenomenon cointegration. He developed methods that have become invaluable in systems where short-run dynamics are affected by large random disturbances and long-run dynamics are restricted to economic equilibrium relationships. Examples include the relations between wealth and consumption, exchange rates and price levels, and short- and long-term interest rates."

Spurious regression.

Standard least-squares method work perfectly well if they are applied to *stationary* time series. But if they are applied to *non-stationary* time series, they can lead to spurious or nonsensical results. One can give examples of two time series that clearly have nothing to do with each other, because they come from quite unrelated contexts, but nevertheless have a high value of R^2. This would normally suggest that a correspondingly high propertion of the variability in one is accounted for by variability in the other – while in fact *none* of the variability is accounted for. This is the phenomenon of *spurious regression*, first identified by G. U. YULE (1871-1851) in 1927, and later studied by Granger and Newbold in 1974. We can largely avoid such pitfalls by restricting attention to stationary time series, as above.

From Granger's obituary (*The Times*, 1.6.2009): "Following Granger's arrival at UCSD in La Jolla, he began the work with his colleague Robert F. Engle for which he is most famous, and for which they received the Bank of Sweden Nobel Memorial Prize in Economic Sciences in 2003. They developed in 1987 the concept of cointegration. Cointegrated series are series that tend to move together, and commonly occur in economics. Engle and Granger gave the example of the price of tomatoes in North and South Carolina .... Cointegration may be used to reduce non-stationary situations to stationary ones, which are much easier to handle statistically and so to make predictions for. This is a matter of great economic importance, as most macroeconomic time series are non-stationary, so temporary disturbances in, say, GDP may have a long-lasting effect, and so a permanent economic cost. The Engle-Granger approach helps to separate out short-term effects, which are random and unpredictable, from long-term effects, which reflect the underlying economics. This is invaluable for macroeconomic policy formulation, on matters such as interest rates, exchange rates, and the relationship between incomes and consumption."

Endogenous and exogenous variables.

The term 'endogenous' means 'generated within'. The $ARCH$ and $GARCH$

models used by Engle and Granger show how variable variance (or volatility) can arise in such a way. By contrast, 'exogenous' means 'generated outside'. Exogenous variables might be the effect in a national economy of international factors, or of the national economy on a specific firm or industrial sector, for example. Often, one has a vector autoregressive (VAR) model, where the vector of variables is partitioned into two components, representing the endogenous and exogenous variables.

Discrete and continuous time.

While econometric data arrives discretely (monthly trade figures, daily closing prices for stocks, etc.), continuous time is more convenient for dynamic models of the economy. See e.g.
A. R. BERGSTROM: *Continuous-time econometric modelling*, OUP, 1990.