ullint4b.tex pm Wed 21.2.2018

5. The change-of-numeraire formula

Here we follow [BM, 2.2]. For more detail, see the paper Brigo & Mercurio (2001c) cited there, and
H. GEMAN, N. El KAROUI and J. C. ROCHET, Changes of numeraire, changes of probability measure and pricing of options. *J. Applied Probability* **32** (1995), 443-458.

We begin our detailed analysis of the market models by deriving the change-of-numeraire formula from (a multivariate version of) Girsanov's theorem. This is worthwhile, as we will use the formula to derive both the LMM and SMM dynamics. Also, this is a general approach that can be used in many asset classes – e.g. credit default swap (CDS) market models; see e.g. [BM, §21.1.2]. We shall also use it to prove the HJM drift condition, which we stated without proof in IV.1.

Recall (MATL480 and Ch. I) that a *numeraire* is any non-dividend-paying tradable asset whose value is always positive. For a numeraire S, the measure $\mathbb{Q}_S$ associated with it is a measure under which Y/S is a martingale for any Y which is the price of a non-dividend-paying tradable asset.

If the numeraire is the bank account – dynamics

$$dB_t = r_t B_t dt$$

– then $\mathbb{Q}_B$ is the classic risk-neutral measure. Indeed, for any asset Y, with dynamics

$$dY_t = \mu_t^B(Y_t)dt + \sigma_t(Y_t)dW_B(t),$$

where W_B is a $\mathbb{Q}_B$-BM,

$$d(Y_t/B_t) = (1/B_t)dT_t + Y_t d(1/B_t)$$

(no Itô-correction term, as B has finite variation (FV), so ordinary – Newton-Leibniz – calculus applies). Now (again as B is FV)

$$d(1/B) = -dB/B^2 = -rdt/B.$$

Combining,

$$d(Y_t/B_t) = \frac{(\mu_t^B(Y_t) - r_t Y_t)}{B_t}dt + \frac{\sigma_t(Y_t)}{B_t}dW_B(t).$$

Now by definition of $\mathbb{Q}_B$, Y/B must be a $\mathbb{Q}_B$-mg. To be a martingale for a regular diffusion process means zero drift. So

$$\mu_t^B(Y_t) - r_t Y_t = 0 : \qquad \mu_t^B(Y_t) = r_t Y_t.$$

This says that $\mathbb{Q}_B$ is the measure under which all non-dividend-paying tradable assets Y have the risk-free rate as growth-rate in the drift. But this is just how the risk-neutral measure $\mathbb{Q}$ is defined (MATL480; I.2). So (unsurprisingly)

$$\mathbb{Q} = \mathbb{Q}_B.$$

But sometimes it is useful to have numeraires other than the bank account B – we shall see examples shortly with the LMM. So we now derive a general formula for changing numeraire, from B to S, say.

Let X be an n-dimensional diffusion process whose dynamics under the measure $\mathbb{Q}_S$ corresponding to numeraire S is given by

$$dX_t = \mu_t^S(X_t)dt + \sigma_t(X_t)CdW_S(t), \qquad W_S \text{ BM under } \mathbb{Q}_S. \qquad (S)$$

Here σ_t is an $n \times n$ square diagnal matrix, with $\mu_t^S(x)$ and $\sigma_t(x)$ deterministic functions of (t, x) that are smooth enough to allow the calculations needed below (sufficient smoothness here is not a significant restriction in practice, so we do not need to go into detail here), and W_S is standard $\mathbb{Q}_S$-BM. The $n \times n$ matrix C is introduced here to model correlation in the resulting driving noise: CdW is equivalent to an n-dimensional BM with instantaneous correlation matrix

$$\rho = CC^T,$$

where the superscript T denotes transposition. (We use a square diffusion matrix for simplicity in using Girsanov's theorem later. This is not in fact necessary, but we do it here for convenience.)

Now suppose we express the dynamics of X in a new numeraire U rather than the old one S. So changing S to U in (S) above,

$$dX_t = \mu_t^U(X_t)dt + \sigma_t(X_t)CdW_U(t), \qquad W_U \text{ BM under } \mathbb{Q}_S. \qquad (U)$$

We can now use Girsanov's theorem to find the Radon-Nikodym (RN) derivative between $\mathbb{Q}_S$ and $\mathbb{Q}_U$ for the X-dynamics under the two different measures. We obtain

$$\zeta_T \;:=\; \frac{d\mathbb{Q}_S}{d\mathbb{Q}_U}|\mathcal{F}_T$$

$$= \exp\{-\frac{1}{2}\int_0^T |(\sigma_t(X_t)C)^{-1}[\mu_t^S(X_t) - \mu_t^U(X_t)]|^2 dt$$
$$+ \int_0^T \Big((\sigma_t(X_t)C)^{-1}[\mu_t^S(X_t) - \mu_t^U(X_t)]\Big)^T dW_U(t)\}.$$

Then ζ is an *exponential martingale* (MATL480 5a, VI.3): setting

$$\alpha_t := [\mu_t^S(X_t) - \mu_t^U(X_t)]^T((\sigma_t(X_t)C)^{-1})^T$$

gives 'exponential martingale dynamics' as the SDE for ζ:

$$d\zeta_t = \alpha_t \zeta_t dW_U(t). \qquad (\zeta : 1)$$

On the other hand, by definition of $\mathbb{Q}_S$, for any tradable asset price Z we have

$$E_0^{\mathbb{Q}_S}[Z_T/S_T] = E_0^{\mathbb{Q}^U}[\frac{U_0 Z_T}{S_0 U_T}],$$

both being equal to Z_0/S_0 (discounted asset prices are mgs, so have constant expectation: on the left, just replace T by 0; on the right, the U-terms are the discounting; the asset is Z/S_0). But by definition of RN derivative, we also have that for all Z,

$$E_0^{\mathbb{Q}_S}[Z_T/S_T] = E_0^{\mathbb{Q}^U}[\frac{Z_T}{S_T}.\frac{d\mathbb{Q}_S}{d\mathbb{Q}_U}].$$

Comparing the two RHSs above, we have that as Z is arbitrary,

$$\zeta_T := \frac{d\mathbb{Q}_S}{d\mathbb{Q}_U}|\mathcal{F}_T = \frac{U_0 S_T}{S_0 U_T},$$

and since ζ is a $\mathbb{Q}_U$-mg,

$$\zeta_t = E_t^{\mathbb{Q}_U}[\zeta_T] = E_t^{\mathbb{Q}_U}[\frac{U_0 S_T}{S_0 U_T}] = \frac{U_0 S_t}{S_0 U_t}. \qquad (*)$$

So differentiating this,

$$d\zeta_t = \frac{U_0}{S_0} d[\frac{S_t}{U_t}].$$

Now S/U is both a numeraire itself, and a $\mathbb{Q}_U$-mg. So it has mg dynamics:

$$d(S_t/U_t) = \sigma_t^{S/U} C dW_U(t) \qquad \text{under } \mathbb{Q}^U.$$

So

$$d\zeta(t) = \frac{U_0}{S_0}.\sigma_t^{S/U} C dW_U(t). \qquad (\zeta : 2)$$

We now have two expressions for $d\zeta_t$, $(\zeta : 1), (\zeta : 2)$. Comparing them,

$$\alpha_t \zeta_t = \frac{U_0}{S_0} \sigma_t^{S/U} C.$$

Substituting for ζ_t here from $(*)$, we obtain

$$\frac{S_t}{U_t} \alpha_t = \sigma_t^{S/U} C.$$

This and the definition above of α_t give the following fundamental result:

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \frac{U_t}{S_t} \sigma_t(X_t) \rho (\sigma_t^{S/U})^T, \qquad \rho = CC^T.$$

This gives the change in the drift of a stochastic process when changing numeraire from S to U (or vice versa).

It often happens that, under the measure $\mathbb{Q}_U$, the S- and U-dynamics are given by SDEs of the form

$$dS_t = (\cdots)dt + \sigma_t^S C dW_U(t),$$

$$dU_t = (\cdots)dt + \sigma_t^U C dW_U(t)$$

(the drifts can be anything here, but if the diffusion terms are any further apart than this, we cannot draw a conclusion). Then (product rule of Itô calculus)

$$d(\frac{S_t}{U_t}) = \frac{1}{U_t} dS_t + S_t d(\frac{1}{U_t}) + dS_t . d(\frac{1}{U_t}),$$

and by Itô's lemma,

$$d(\frac{1}{U_t}) = -\frac{1}{U_t^2} dU_t + \frac{1}{U_t^3} (dU_t)^2.$$

Combining, and retaining only dW^U terms (so neglect terms in $(dt)^2$, $(dW_U)^2$, as always in Itô calculus),

$$d(S/U) = (...)dt + \Big(\frac{\sigma^S}{U} - \frac{S}{U}\frac{\sigma^U}{U}\Big) C dW_u.$$

This identifies the diffusion coefficient of the numeraire S/U:

$$\sigma_t^{S/U} = \frac{\sigma_t^S}{U_t} - \frac{S_t}{U_t}\frac{\sigma_t^U}{U_t}.$$

Substituting this in the result above:

Theorem (Change-of-numeraire formula). Under the above circumstances,

$$\mu_t^U(X_t) = \mu_t^S(X_t) - \frac{U_t}{S_t}\sigma_t(X_t)\rho\Big(\frac{\sigma_t^S}{U_t} - \frac{S_t}{U_t}\frac{\sigma_t^U}{U_t}\Big)^T, \qquad \rho = CC^T.$$

Shocks.

It is sometimes helpful to consider what happens in terms of "shocks". Equating the expressions for dX_t in (S) and (U) above,

$$\mu_t^U(X_t)dt + \sigma_t(X_t)CdW_U(t) = \mu_t^U(X_t)dt + \sigma_t(X_t)CdW_U(t).$$

Substituting in the Theorem above gives

$$CdW_S(t) = CdW_U(t) - \rho\Big(\frac{\sigma_t^S}{U_t} - \frac{S_t}{U_t}\frac{\sigma_t^U}{U_t}\Big)^T dt.$$

If we abbreviate the notation by writing the vector diffusion coefficient of a diffusion X by $DC(X)$, and we write the correlated Brownian motion as

$$dZ = CdW,$$

the above becomes

$$dZ_S(t) = dZ_U(t) - \rho\Big(\frac{DC(S)}{S_t} - \frac{DC(U)}{U_t}\Big)^T dt \qquad (CBM)$$

((CBM) here stands for *correlated Brownian motion*).

Below, we will apply the change-of-numeraire technique to three things:
(i) Black's caplet formula;
(ii Black's swaption formula;
(iii) the Heath-Jarrow-Morton drift condition.

6. LMM (LIBOR Market Model) dynamics

We can use the results above to give a rigorous proof of Black's caplet formula of 1976 (the techniques above came much later). In III.5 above, take (with $P(t,T)$ the bond prices as before, giving $P(.,T) = (t \mapsto P(t,T))$ as a function of t)

$$U = P(.,T_i), \qquad \mathbb{Q}_U = \mathbb{Q}_i.$$

Since

$$F(t;T_{i-1},T_i) = (1/\tau_i)\frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)},$$

$$F_i(t) := F(t;T_{i-1},T_i)$$

is a $\mathbb{Q}_i$-mg. Take

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i(t), \qquad \mathbb{Q}_i, \qquad t \le T_{i-1}$$

(notation as above). This is the LIBOR Market Model (LMM) (this is the common name; Brigo and Mercurio [BM, 6.2, p.202] prefer the name **L**ognormal Forward-LIBOR **M**odel (LMM).

One time-interval.

Consider the discounted T_{k-1}-caplet

$$(F_k(T_{k-1}) - K)_+ B(0)/B(T_k).$$

With $E_k[.]$ for $\mathbb{Q}_k$-expectation, the time-0 price of the caplet is, by FACT 2 (V.1)

$$\begin{aligned} B(0)E_{\mathbb{Q}_B}[(F_k(T_{k-1}) - K)_+/B(T_k)] &= P(0,T_k)E_k[(F_k(T_{k-1}) - K)_+/P(T_k,T_k)] \\ &= P(0,T_k)B\&S(F_k(0),K,v_k\sqrt{T_{k-1}}), \end{aligned}$$

where we write $B\&S(.)$ for the Black-Scholes formula for calls, of which Black's caplet formula is clearly a variant (arguments: initial stock price, strike, volatility), and

$$v_k := \frac{1}{T_{k-1}}\int_0^{T_{k-1}} \sigma_k(t)^2 dt.$$

The dynamics of F_k is easy under $\mathbb{Q}_k$. But if we price a product depending on several forward rates at the same time, we need to fix a pricing measure, say $\mathbb{Q}_i$, and model all rates F_k under this same measure $\mathbb{Q}_i$. This is handled as above for $k = i$, but not when $i < k$ or $i > k$ (below).

Black volatility.

The v_k above is a volatility as in the Black-Scholes formula, and the caplet price above is an option price (on an interest rate, rather than a stock as in Black-Scholes). Recall (MATL480) that (with vega the partial derivative of the option price wrt volatility) *vega is positive* ("options like volatility"). So (as a continuous strictly increasing function has a well-defined inverse function) there is a one-one correspondence between option prices and volatilities, and one can go back and forth between the two, i.e. obtain either from the other. We can *see* the prices at which options are traded in the market; the corresponding volatility is the *implied volatility*. The same applies here. Traders in caplets speak of the v_k above, obtained as an implied volatility in this way, as the Black volatility, or *Black vol* for short. They have a very well-developed intuition for it (as stock-market traders do for implied vol there): *this is the way traders think*. See e.g. [BM, p.197, p.287-288].

Several time-intervals.

We are now going to handle the $i < k$ and $i > k$ cases left open above by the change-of-numeraire toolkit of V.5 above.

$i < k$.

We use (CBM) from V.5 above:

$$dZ_S(t) = dZ_U(t) - \rho\Big(\frac{DC(S)}{S_t} - \frac{DC(U)}{U_t}\Big)^T dt. \qquad (CBM)$$

Here DC is a linear operator on diffusions: $DC(X_t)$ is the row-vector $\mathbf{v}$ in

$$dX_t = (\cdots)dt + \mathbf{v}dZ_t,$$

for diffusion processes X describable in terms of a common column-vector of driving noise, a vector BM Z. So if

$$dF_1 = \sigma_1 F_1 dZ_1,$$

then

$$DC(F_1) = (\sigma_1 F_1, 0, \cdots, 0) = \sigma_1 F_1 e_1,$$

say. The correlation matrix ρ is the instantaneous correlation between the shocks (the same under any measure),

$$dZ_i dZ_j = \rho_{ij} dt.$$

The toolkit (CBM) above can also be written

$$dZ_S(t) = dZ_U(t) - \rho(DC(\log(S/U)))^T dt. \qquad (CBM^*)$$

For,

$$\begin{aligned} \frac{DC(S)}{S} - \frac{DC(U)}{U} &= DC(\log S) - DC(\log U) \\ &= DC(\log S - \log U) \\ &= DC(\log(S/U)). \end{aligned}$$

We now apply the toolkit: taking $S = P(., T_k)$ and $U = P(., T_i)$, (CBM^*) gives

$$dZ_k(t) = dZ_i(t) - \rho(DC(\log(P(., T_k)/P(., T_i)))^T dt.$$

Now by (F_j) (V.1, W4a),

$$\begin{aligned} \log\Big(\frac{P(t,T_k)}{P(t,T_i)}\Big) &= \log\Big(\frac{P(t,T_k)}{P(t,T_{k-1})} \frac{P(t,T_{k-1})}{P(t,T_{k-2})} \cdots \frac{P(t,T_{i+1})}{P(t,T_i)}\Big) \\ &= \log\Big(\frac{1}{1+\tau_k F_k(t)} . \frac{1}{1+\tau_{k-1}F_{k-1}(t)} \cdots . \frac{1}{1+\tau_{i+1}F_{i+1}(t)}\Big) \\ &= \log\Big(1/[\prod_{j=i+1}^{k} (1+\tau_j F_j(t))]\Big) \\ &= -\sum_{j=i+1}^{k} \log(1+\tau_j F_j(t)). \end{aligned}$$

So linearity of DC gives

$$\begin{aligned} DC\log\Big(\frac{P(t,T_k)}{P(t,T_i)}\Big) &= -\sum_{j=i+1}^{k} DC\log(1+\tau_j F_j(t)) \\ &= -\sum_{j=i+1}^{k} \frac{DC(1+\tau_j F_j(t))}{1+\tau_j F_j(t)} \end{aligned}$$

(as in the calculation for DC above)

$$= -\sum_{j=i+1}^{k} \tau_j \frac{DC(F_j(t))}{1+\tau_j F_j(t)}$$

$$= -\sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j}{1+\tau_j F_j(t)},$$

with e_j the row-vector (δ_{ij}) (Kronecker delta – 1 in the jth position, 0 elsewhere). Combining,

$$dZ_k(t) = dZ_i(t) + \rho \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j}{1+\tau_j F_j(t)} dt.$$

Pre-multiply both sides by e_k. We obtain

$$dZ_k^k = dZ_i^k + [\rho_{k1}, \rho_{k2}, \cdots, \rho_{kn}] \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) e_j}{1+\tau_j F_j(t)} dt,$$

in an obvious notation (the superscripts k here denote evaluation at time T_k)

$$= dZ_i^k + \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) \rho_{kj}}{1+\tau_j F_j(t)} dt.$$

Substitute this in the dynamics written in the above notation,

$$dF_k = \sigma_k F_k dZ_k^k$$

to obtain

$$dF_k = \sigma_k F_k \Big(dZ_i^k + \sum_{j=i+1}^{k} \tau_j \frac{\sigma_j(t) F_j(t) \rho_{kj}}{1+\tau_j F_j(t)} dt \Big).$$

This finally gives the dynamics of a forward rate with maturity k under the forward measure with maturity $i < k$. The case $i > k$ is handled similarly.

Write the drifts here as

$$\mu_i^m := \sum_{j=i+1}^{m} \tau_j \frac{\sigma_j(t) F_j(t) \rho_{mj}}{1+\tau_j F_j(t)}.$$

Then the above becomes:

$$dF_k(t) = \mu_i^k(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t) \qquad (i < k),$$

$$dF_k(t) = \sigma_k(t)F_k(t)dZ_k^k(t) \qquad (i = k),$$

$$dF_k(t) = -\mu_k^i(t, F(t))\sigma_k(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^i(t) \qquad (i > k).$$

These SDEs may be shown to have a unique solution. We omit the details; see [BM, 6.3.2].

Black's swaption formula; the swaption market model (SMM).
See e.g. [BM, 6.7, 6.13-17], and for hedging, especially [BM 6.7.1].

Incompatibility between LMM and SMM
We refer to [BM, 6.8] for details. We have already mentioned the result, and will feel free to use it. This incompatibility is not a serious problem in practice, as the two give results in good agreement.

Note. The analogy with Physics may be useful. The two great advances in Physics in the 20th century were Quantum Theory (dealing with the very small – subatomic particles, etc.), and Einstein's General Theory of Relativity (dealing with the very large – cosmology, galaxies etc.). We know that each is right rather than wrong. We also know that the two are *incompatible*. The search for a Grand Unified Theory (to unify the four fundamental forces of Nature – gravity [in relativity] with electromagnetism and the weak and strong nuclear forces [in quantum theory]) is motivated by this. We do not know whether this search will ever succeed; meanwhile Physics goes on, using different methods in different contexts. Similarly here.

7. The Heath-Jarrow-Morton (HJM) drift condition

We discussed earlier (IV.1) the HJM framework for the forward rates $f(t, T)$. While we take the view that most useful models are for r (Ch. II) or F_i, S_{ij} (Ch. V), HJM is still important, in a number of areas (commodities, etc.) and historically. We stated the HJM drift condition earlier without proof; we now have the tools to prove it, so we do so. Recall that under the risk-neutral measure $\mathbb{Q}$ with bank account B as numeraire,

$$df(t,T) = \sigma(t,T)(\int_t^T \sigma^T(t,s)ds)dt + \sigma(t,T)dW_B(t) \qquad (HJM)$$

as needed to give no arbitrage (NA) – which we need. That is, to avoid arbitrage, *the drift is completely determined by the volatilities.* We work in n dimensions, with σ a row n-vector and W a column n-vector BM. Correlations will be present, but we put them in the inner product $\sigma\sigma^T$ rather than in the BM W (recall the correlated BMs involving C in V.6 above). We use the change-of-numeraire technique.

Recall that

$$f(t,T) = -\frac{1}{P(t,T)}\frac{\partial P(t,T)}{\partial T} \sim \frac{P(t,T) - P(t,T+\Delta T)}{P(t,T)\Delta T}$$

for small ΔT. So this is a tradable asset (difference of two bonds) divided by a second asset (the bond $P(t,T)$), and by FACT 1 of V.1 it is a mg under the $P(.,T)$ numeraire measure $\mathbb{Q}_T$, which we call *T-forward measure.* Since a mg has zero drift,

$$df(t,T) = \sigma(t,T)dW_T(t)$$

under the T-forward measure. Now use the change-of-numeraire toolkit – formula (CBM^*) of V.6 above. As Z here is W, which has independent components (above), the Brownian covariance matrix here is the identity:

$$\rho = I.$$

We choose numeraires $S = B$ (bank account) and $U = P(.,T)$. Then

$$dW_B(t) = dW_T(t) - (DC(\log(B/P(.,T))))^T dt.$$

As before,

$$DC(\log(B/P(.,T)) = DC(\log B) - DC(\log P(.,T)) = -DC(\log P(.,T)).$$

So we now need to find $DC(\log P(.,T))$.

Integrating the definition

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T},$$

$$P(t,T) = \exp\{-\int_t^T f(t,u)du\} : \qquad \log P(t,T) = -\int_t^T f(t,u)du.$$

Differentiate wrt t:

$$d_t \log P(t,T) = f(t,t)dt - \int_t^T d_t f(t,u)du = -\int_t^T [(\cdots)dt + \sigma(t,u)dW_t]du,$$

whichever measure we are in, provided $\sigma(t,u)$ is the vector volatility for $df(t,u)$. This SDE gives the diffusion coefficient of $d\log P(t,T)$ as

$$DC(\log P(.,T)) = -\int_t^T \sigma(t,u)du.$$

As the bank-account numeraire B has no diffusion coefficient, this gives

$$DC(\log(B/P(.,T)) = \int_t^T \sigma(t,u)du.$$

So (CBM^*) (V.6) gives

$$dW_B(t) = dW_T(t) - \int_t^T \sigma(t,u)du : \qquad dW_T(t) = dW_B(t) + \int_t^T \sigma(t,u)du.$$

Substituting this into our initial SDE

$$df(t,T) = \sigma(t,T)dW_T(t)$$

gives

$$\begin{aligned} df(t,T) &= \sigma(t,T)[dW_B(t) + (\int_t^T \sigma^T(t,u)du)dt] \\ &= \sigma(t,T)(\int_t^T \sigma^T(t,u)du)dt + \sigma(t,T)dW_B(t), \end{aligned}$$

giving the HJM drift condition, as required. //

Note. Models developed according to the general HJM framework are often non-Markovian, and can even be infinite-dimensional. But if the volatility structure of the forward rates satisfy certain conditions, then an HJM model can be expressed entirely by a finite-state Markov chain, making it computationally feasible. Examples include a one-factor, two state model:
O. Cheyette, Term Structure Dynamics and Mortgage Valuation, *J. Fixed Income* 1, 1992;
P. Ritchken and L. Sankarasubramanian, Volatility Structures of Forward Rates and the Dynamics of Term Structure, *Math. Finance* 5, 1995),
and later multi-factor versions.