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Breeden-Litzenberger formula

The curve

$$K \mapsto v_2^{Mkt}(T_1, K)/\sqrt{T_1}$$

is called the *volatility smile* of the T_1-expiry caplet. Just as with the Black-Scholes formula, this curve should be flat if Black's formula were exact. But instead, we see smile shapes, or 'skew smiles' – 'smirks'.

Clearly, only some strikes K are quoted in the market, so the remaining points have to be obtained by, say, interpolation, or by using some other model. For fixed expiry T_1, interpolation in K can easily be done, but gives no insight into the underlying forward-rate dynamics compatible with such prices.

Let p_2 be the density of $F_2(T_1)$ under the T_2-forward measure (if Black's formula were exact, this density would be lognormal). As the caplet is an option,

$$\begin{aligned}
Cpl(0, T_1, T_2, K) &= P(0, T_2)\tau Bl(K, F_2(0), v_2(T_1)) \\
&= P(0, T_2)\tau E_0^2[(F(T_1, T_1, T_2) - K)_+] \\
&= P(0, T_2)\tau \int (x - K)_+ p_2(x) dx.
\end{aligned}$$

Consider now the possibility of *differentiating* this with respect to K. We assume that p_2 is smooth enough to justify *differentiating under the integral sign* – interchanging $\partial/\partial K$ and $\int ..dx$. As

$$(\partial/\partial K)[(x - K)_+] = -I(K < x)$$

(the derivative does not exist at the point x, but as this point contributes nothing to $\int ...dx$ this makes no difference), this gives

$$\begin{aligned}
\frac{\partial}{\partial K} Cpl(0, T_1, T_2, K) &= P(0, T_2)\tau \int -I(K < x) p_2(x) dx \\
&= -P(0, T_2)\tau \int_K^\infty p_2(x) dx.
\end{aligned}$$

We can now differentiate both sides to obtain:

Theorem (Breeden-Litzenberger formula (1978)). Under the above smoothness condition, the density $p_2(K)$ is given by the *second partial derivative of the caplet price w.r.t. the strike* K:

$$p_2(K) = \frac{\partial^2}{\partial K^2} Cpl(0, T_1, T_2, K)/(\tau P(0, T_2)). \qquad (BL)$$

So in principle, we can use the Breeden-Litzenberger formula to pick up the density p_2 from caplet prices. But in practice, this is fraught with difficulties. First, caplets are not traded in the market for *all* strikes K, but only for *some*, $\{K_1, \cdots, K_n\}$ say. So we have to use what we have – observed caplet prices at *these* strikes – to *interpolate* to obtain a function approximating, or representing, caplet prices for all K. Now interpolation is a *numerical* procedure. It would probably be done in practice by use of *cubic splines* (piecewise-cubic curves whose values, and those of the first two derivatives, are continuous across the points K_i, called the *knots*); for background here, see e.g.
[BF] N. H. BINGHAM and John M. FRY, *Regression: Linear models in statistics*, Springer Undergraduate Mathematics Series (SUMS), 2010, p.212.
But however we interpolate, this has to be done *numerically*. We then have to *differentiate* the result, again numerically. This is dangerous: differentiation is an *unsmoothing process*: it *magnifies* numerical errors in the data. Worse: we then have to differentiate *again*. The upshot is that, while the Breeden-Litzenberger formula is very nice to have, it is of very limited use in practice.

Further: when we have the density p_2 (overlooking the numerical inaccuracies inherent in it – see above): what kind of F-dynamics does it come from?

Just as the finite-difference approximation to a derivative is a difference quotient, the finite-difference approximation to a second derivative is of the form

$$\frac{\partial^2}{\partial K^2} C \sim [C(K + \Delta K) - 2C(K) + C(K - \Delta K)]/(\Delta K)^2.$$

For this, we would need prices of C for three nearby strikes, K, $K \pm \Delta K$. Similarly for Dupire's formula, below.

For background here, see a book on Numerical Analysis. The relevant subject here is the *Calculus of Finite Differences* (the name is in contrast to the old name for (ordinary) calculus – *Infinitesimal Calculus*).

Dupire's formula

Suppose we have an option on the forward rate $F(T)$, with payoff function h and expiry T. For $t \in [0, T]$, if

$$v(t, x) := E[h(F_T)|F_t = x],$$

$$\begin{aligned} E[h(F_T)] &= E[E[h(F_T)|F_t = x]] \qquad \text{(tower property)} \\ &= \int_0^\infty v(t, x)\phi(t, x)dx, \end{aligned}$$

if F_t has density $\phi(t, x)$. Now the LHS is *independent* of t. Hence, so too is the RHS: differentiating under the integral sign w.r.t. t as above,

$$0 = \int \frac{\partial v}{\partial t}\phi dx + \int v\frac{\partial \phi}{\partial t}dx.$$

Now, v satisfies the*Kolmogorov backward equation*, also known as the *Fokker-Planck equation*:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma(t, x)^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \qquad v(T, x) = h(x). \qquad (FoPl)$$

This belongs to the theory of *diffusion equations*, which we touched on in MATL480, V.2.4. We quote it here; for background, see any good book on probability and stochastic processes (or Google 'Fokker-Planck'). Using $(FoPl)$, we can substitute for the $\partial v/\partial t$ term in the above, to obtain (writing v' for $\partial v/\partial x$, etc.)

$$0 = -\frac{1}{2}\int (\sigma^2 x^2 \phi)v'' dx + \int v\frac{\partial \phi}{\partial t}dx. \qquad (*)$$

Integrate the first integral by parts: the integrated term vanishes (at 0 because of the x^2, at infinity because the other factors decay fast enough – we quote this too), giving

$$\int (\sigma^2 x^2 \phi)v'' dx = \int (\sigma^2 x^2 \phi)dv' = -\int (\sigma^2 x^2 \phi)' v' dx = -\int (\sigma^2 x^2 \phi)' dv.$$

Integrate by parts again: again the integrated terms vanish, giving

$$\int (\sigma^2 x^2 \phi)dv = \int v(\sigma^2 x^2 \phi)'' dx.$$

Substituting this in $(*)$,

$$0 = \int (\frac{1}{2}(\sigma^2 x^2 \phi - \frac{\partial \phi}{\partial \tau}) v dx.$$

But the payoff h, and so the conditional density v, is arbitrary. So the integrand here must vanish, giving the *forward equation* (so called because it deals with forward rates, III),

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma(t,x)^2 x^2 \phi). \qquad (ForEq)$$

Note. 1. The Kolmogorov forward and backward equations are PDEs for the transition densities $p(t,x)$ of diffusion equations, obtained by considering time-intervals $(t, t+dt)$ (looking forwards from t), $(t-dt, t)$ (looking backwards from t) respectively.
2. The proof of $(ForEq)$ above is substantially that of the Kolmogorov forward and backward equations (so we have not omitted very much).
3. We do not need to worry here about the distinction between the Kolmogorov forward and backward equations – certainly not in one dimension. One-dimensional diffusion equations are *time-reversible.*

We do, of course, have to take notice of the 'forward' in the forward rate $F(T)$ here (cf. Ch. IV, Forward-rate models).

Suppose now that the option above is a call C with strike K. Then

$$C(T,K) = E[(F-K)_+] = E[(F-K)I(F>K)] = \int_K^\infty (x-K)\phi(t,x)dx.$$

So, first differentiating under the integral sign w.r.t. K (as above),

$$\partial C(T,K)/\partial K = -\int_K^\infty \phi(T,x)dx$$

(the $(x-K)$ term vanishes at the lower limit). So

$$\partial^2 C(T,K)/\partial K^2 = \phi(T,K). \qquad (**)$$

Next, differentiate w.r.t. T (under the integral sign on the right, as above) and use $(ForEq)$:

$$\frac{\partial C(T,K)}{\partial T} \quad = \quad \int_K^\infty (x-K) \frac{\partial \phi(T,x)}{\partial T} dx$$

$$
\begin{aligned}
&= \int_K^\infty (x-K).\frac{1}{2}(\sigma^2 x^2 \phi)'' dx \qquad \text{(by } (ForEq)) \\
&= -\frac{1}{2}\int_K^\infty (\sigma^2 x^2 \phi)' dx = -\frac{1}{2}\int_K^\infty d(\sigma^2 x^2 \phi) \qquad \text{(integrating by parts)} \\
&= \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K) \qquad \text{(lower limit, hence the -)},
\end{aligned}
$$

performing the integration. This gives, by $(**)$:

Theorem (Dupire's formula). In the notation above, the call price satisfies

$$C(T,K) = \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K).$$

That is, the *local volatility* $\sigma(T,K)$ is completely specified by the *volatility surface* $\sigma(K,T)$ (via its derivatives) by Dupire's formula,

$$\sigma(T,K) = \frac{1}{K}\sqrt{\frac{2\partial C(T,K)/\partial T}{\partial^2 C(T,K)/\partial K^2}}. \qquad (Dup)$$

Application to caplets

There is a problem in applying Dupire's formula to the caplet market. There, we do not have a continuum of traded maturities for options on the forward rate F_2, as we noted above. The only instant of interest in a forward rate is typically its reset date T_1, since it then becomes a LIBOR rate. And payoffs contain LIBOR rates, not Forward-LIBOR rates. We might have caplets on:

$L(T_1,T_2) = F_2(T_1)$, maturity T_2,
$L(T_2,T_3) = F_3(T_2)$, maturity T_3,
$L(T_3,T_4) = F_4(T_3)$, maturity T_4, etc.

But the forward rates involved are different, so we cannot assume that we have options on several maturities $T_2, T_3, T_4, \cdots$ for the same F, as Dupire's method would require. Dupire's method does work when the asset is always the same, as in the equity (stock) or FX (Forex, foreign-exchange) markets.

Dupire's method is in fact non-parametric (see e.g. NHB, SMF, Ch. VI), since it aims to derive the diffusion coefficient (the volatility here) as a function of the whole market surface (in maturity and strike).

But we need only work in the strike dimension, since maturity is fixed for

a caplet. We can then proceed the other way round – a parametric approach: Assume dynamics a priori, depending on given parameters.
Price options with the right maturity and these dynamics.
These prices will depend on the parameters.
Set the parameters so as to match the relevant option prices observed in the market for this maturity.

Start from the parametric dynamics

$$dF(t; T_1, T_2) = \nu(t, F(t; T_1, T_2))dW(t).$$

This generates a smile; see below.

Here the dynamics, given by the function ν, can be either deterministic or stochastic. In the second case, we have a *stochastic volatility (SV)* model; e.g.,

$$\nu(t, F) = \sqrt{\xi(t)}F, \qquad d\xi(t) = \kappa(\theta - \xi(t)) + \eta\sqrt{\xi(t)}dZ(t), \qquad (SV)$$

$$dZdW = \rho_{W,Z}dt.$$

We concentrate here on a deterministic ν, giving a *local volatility (LV)* model, e.g.

$$\nu(t, F) = \sigma_2(t)F^{\gamma} \qquad (0 \leq \gamma \leq 1), \qquad (CEV)$$

with σ_2 deterministic. Here CEV stands for *constant elasticity of volatility.*

One problem with local volatility models is that they tend to flatten the smile in the future, conditioned on future information. For example, think of some future time $u > 0$, where we consider the smile for maturity $u + T$ given what we know at time u. We have no reason to expect this to be any flatter (or different in any other way) from the smile at time T given what we know now at time 0. Now LV models do flatten the smile in this way, while SV models do not; this is an important advantage of SV over LV.

To summarise:
1. The true forward-adjusted density p_2 of F_2 is linked to caplet (Call on F) and floorlet (Put on F) market prices through second-order differentiation wrt strikes (Dupire's formula).
2. We need the dF dynamics to be as compatible as possible with the density p_2.
3. Dupire's method works on ps extracted from prices by interpolation, rather than on prices directly, and then obtains dF based on this. But the interpolation interferes strongly with the result, and the method is unstable.

4. One can instead parametrise dF and fit the prices this parametrisation implies to the market caplet prices $Cpl^{Mkt}(0, T_1, T_2, K_i)$ for the strikes K_i quoted in the market.
5. But this parametrisation has to be flexible, and has to lead to a tractable model, in order to be useful.
6. Finally, we have to deal in general with an implied-volatility surface, since we have a caplet-volatility curve for each expiry. The calibration issues are as before, except for the larger computational effort as the size of the data set increases.

Shifted lognormal (displaced diffusion) model for smile

Assume that the forward rate F_j evolves under its associated T_j-forward measure according to the dynamics

$$F_j(t) = X_j(t) + \sigma, \qquad dX_j(t) = \beta(t) X_j(t) dW(t),$$

with α a real constant, β a deterministic function of time and W BM:

$$dF_j(t) = \beta(t)(F_j(t) - \alpha) dW(t)$$

(a shifted form of geometric Brownian motion, GBM). Then $F_j(T)$, conditional on $F_j(t)$, $t < T \leq T_{j-1}$, is a shifted lognormal. We retain the analytic tractability of GBM:

$$E_t^j[(F_j(T_{j-1} - K)_+] = E_t^j[(X_j(T_{j-1} - (K - \alpha))_+],$$

so that for $\alpha < K$ the caplet price is given by Black's formula:

$$Cpl(t, T_{j-1}, T_j, K) = \tau P(t, T_j) Bl\Big(K - \alpha, F_j(t) - \alpha, (\int_t^{T_{j-1}} \beta(u)^2 du)^{\frac{1}{2}}\Big).$$

The implied Black volatility

$$\hat{v}/\sqrt{T_{j-1}} = \hat{v}(K, \alpha)/\sqrt{T_{j-1}}$$

(at $t = 0$, say) is obtained by 'backing out' the volatility parameter $\hat{v}$ in Black's formula that matches the model price (the term 'backing out', and this procedure, we have met before, in terms of *implied volatility* in the Black-Scholes formula, MATL480, VI.2):

$$Bl(K, F, \hat{v}(K, \alpha) = Bl\Big(K - \alpha, F_j(t) - \alpha, (\int_t^{T_{j-1}} \beta(u)^2 du)^{\frac{1}{2}}\Big), \qquad F = F_j(0).$$

The role of α.

For $\alpha = 0$, the implied caplet volatility is flat (constant). First, for α non-zero, the curve is strictly decreasing ($\alpha < 0$) or increasing ($\alpha > 0$). Second, it moves the curve upwards ($\alpha < 0$) or downwards ($\alpha > 0$):

increasing/decreasing α shifts the volatility curve $K \mapsto \hat{v}(K, \alpha)$ down/up. For the best fit, one often needs *decreasing* implied volatility curves, which correspond to negative α, and so to negative values in the support of the forward-rate density (i.e., the possibility of negative values of this density). Even though the probability of such events is small in practice, the possibility of them is regarded as an undesirable feature.

The CEV model.

This is due to J. C. Cox (1975) and S. E. Ross (1976) (both of the Cox-Ross-Rubinstein (binomial tree) model: MATL480, IV.5). As above,

$$dF_j(t) = \sigma_2(t)[F_j(t)]^\gamma dW(t) \qquad (0 \leq \gamma \leq 1), \qquad (CEV)$$

$$F_j = 0 \text{ is an absorbing boundary when } 0 < \gamma < \tfrac{1}{2}$$

(this SDE does not have a unique solution unless we impose the boundary condition when $0 < \gamma < \frac{1}{2}$).

The time-dependence of σ_j can be dealt with by a deterministic time change. Setting

$$v(\tau, T) = \int_\tau^T \sigma_j(s)^2 ds, \qquad \tilde{W}(v(0,t)) := \int_0^t \sigma_j(s) dW(s),$$

we obtain a BM $\tilde{W}$ with time-parameter v. We make this time change in the SDE above by setting

$$f_j(v(t)) := F_j(t) :$$

$$df_j(v) = f_j(v)^\gamma d\tilde{W}(v)$$

(with the boundary condition as before). This can be transformed into the SDE for a *Bessel process* via a change of variables. Bessel processes are among the most well-known and tractable diffusions. Appealing to their theory, we quote: the transition density of $F_j(T)$ conditional on $F_j(t)$, $t < T \leq T_{j-1}$, is non-central chi-squared.

Hence an explicit formula can be given for the caplet price [BM, 10.2].

14. Is volatility rough? Fractional Brownian motion

[GJR] Jim GATHERAL, Thibault JAISSON and Mathieu ROSENBAUM, Volatility is rough. arXiv:1410.3394v1.

In this influential paper, the authors use fractional Brownian motion (fBM) to model volatility; fBM is parametrised by the *Hurst parameter* H; $H = \frac{1}{2}$ gives BM; $H < \frac{1}{2}$ gives volatility *rougher* than BM (and $H > \frac{1}{2}$ gives smoother).

Question: Is volatility rough?

The authors conclude that it is. They further conclude that the reasons volatility is rough are mainly two-fold: order-splitting and high-frequency trading. The influence of the second is clear. For the first: order-splitting is the process of taking a – possibly large – order, splitting it, and executing the resulting sub-orders separately. One motivation for this concerns the detailed mechanism by which markets match asks (demand – buyers; as low a price as possible) to bids (supply – sellers; as high a price as possible). The area here is *limit orders*; we must refer elsewhere for this. Another motivation is to keep trades small, so as not to shift prices against one.

The fractional BM above is the Gaussian process with mean 0 and covariance

$$C_H(s,t) := E[X_s X_t] = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

This is indeed a Gaussian covariance (we quote this). One can check that for $H = \frac{1}{2}$ this reduces to the familiar covariance $\min(s,t)$ of standard Brownian motion.

The H here is for Hurst. Hurst was a hydrologist; the process originates in his work on the flow of water in the River Nile. Fractional BM is widely used to model *long memory*. This too is motivated by the Nile. Recall the biblical passage, when the Israelites were enslaved in Egypt under the pharoahs in the time of Joseph, about seven fat years followed by seven lean years. This *Joseph effect* is an example of long memory.

Of course, in our subject, we have an example of long memory in the failure of the world economy to recover from the Crash, even given a decade and unprecedented measures – very low interest rates, QE etc. (VI.6) – designed to stimulate or kick-start the economy, to get it back to its previous (by contrast, very desirable) condition.

Fractional BM has a stochastic (Itô) calculus, reducing to the one we know for BM when $H = \frac{1}{2}$.

15. Other models

1. SABR

The SABR (pronounced 'sabre') model is an SV model introduced by Hagan, Kumar, Lesniewski and Woodward in 2002. The acronym SABR stands for *stochastic alpha beta rho*, for three of the four parameters it contains (below). Under SABR, the forward rate F_j evolves under its associated measure $\mathbb{Q}_j$ by the dynamics

$$\begin{aligned} dF_j(t) &= V(t)[F_j(t)]^{\beta} dZ_j^j(t), \\ dV(t) &= cV(t)dW^j(t), \\ V(0) &= \alpha, \end{aligned}$$

where Z_j^j and W_j are $\mathbb{Q}_j$-standard BMs, correlated by

$$dZ_j^j(t)dW_j(t) = \rho dt.$$

The model is widely used in practice because of its simplicity and tractability – but, it can be problematic! For details, see e.g. [BM, 11.4].

2. Flesaker-Hughston

This model, dating frm 1996, was one of the first to go beyond the short-rate framework on II. It deals with the *state-price densities – pricing kernels* of I.2. It has advantages, particularly with exchange rates and in dealing with interest-rate curves in different currencies. But it has problems. For details, we refer to [BM, A.3].

3. Rogers

This approach, due to L. C. G. Rogers (1997), has as one of its advantages that it gives *positive* interest rates. This seemed highly desirable and natural in 1997, pre-Crash; now that negative interest rates have been observed (I.5), this seems less important. It owes its name to the *Riesz decomposition* (Marcel Riesz (1886-1969) in 1937-38). This originates in pure mathematics; in its probabilistic form, it reads: if X is a uniformly integrable (UI) supermartingale, it has a unique (Riesz) decomposition into

$$X = Y + Z,$$

where Y is a UI mg and Z is a *potential*: that is, a non-negative UI supermg tending to zero at infinity.

Regarding the name: potential theory has its roots in classical physics: the potential of Newtonian (gravitational) attraction, and electromagnetic potential; these have deep similarities, as both obey an Inverse Square Law. Following the discovery (by Kakutani in 1944) of deep connections between potential theory (which had by then entered pure mathematics, in particular complex analysis) and Brownian motion (MATL480, V.2.4), the new field of probabilistic potential theory emerged.

For a summary, see [BM, A.4]. One starts with a Markov process X, and a positive function f. Recall that the state-price density ζ_t is the reciprocal of the chosen numeraire, $1/\zeta_t$. One needs the *resolvent* R_α of X, and its *generator* G (we will have to leave these terms undefined here). Rogers sets

$$\zeta_t = e^{-\alpha t} R_\alpha (\alpha - G) f,$$

and obtains the short rate as

$$r_t = [(\alpha - G) f(X_t)]/f(X_t)$$

(note that this is positive, as f is!) This approach is well suited to modelling interest-rate curves in different currencies.

4. Brody-Hughston

[BH] D. BRODY and L. P. HUGHSTON, Chaos and coherence: a new framework for interest-rate modelling. *Proc. Royal Soc. A* **460** (2004), 85-110.

The authors (this is the same Hughston as above! – both former colleagues of mine at Imperial) both have a background in physics. In particular, they use the formalism of *Wiener chaos expansions*. These are widely used in quantum field theory (QFT) – Fock space, Wick expansions etc. – and in probability and stochastic analysis – iterated Wiener integrals. The method expands the infinite-dimensional objects we encounter in interest-rate theory; these expansions can be truncated, the accuracy of the approximation reflecting the number of terms retained. We must refer to [BH] for details.