ullintexamsoln17.tex

MATL481 INTEREST RATE THEORY: EXAM SOLUTIONS 2017

Q1. *Rho.*
(i) *Rho for calls.*

With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$, $\tau := T - t$ the time to expiry, the Black-Scholes call price is, with d_1, d_2 as given,

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \qquad (BS)$$

So as $d_2 = d_1 - \sigma\sqrt{\tau}$,

$$\phi(d_2) = \frac{e^{-\frac{1}{2}(d_1-\sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*)$$

Differentiating (BS) partially w.r.t. r gives, by $(*)$,

$$\begin{aligned} \rho := \partial C/\partial r &= S\phi(d_1)\partial d_1/\partial r - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(d_1 - d_2)/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial r + K\tau e^{-r\tau}\Phi(d_2) = K\tau e^{-r\tau}\Phi(d_2) : \end{aligned}$$

$$\rho > 0. \qquad [\mathbf{7}]$$

(ii) *Financial interpretation.*

As r increases, cash becomes more attractive compared to stock. So stock *buyers* have a 'buyer's market', favouring them. So for *calls* (options to buy), $\rho > 0$. **[3]**

(iii) *Rho for puts.*

By put-call parity, $S + P - C = Ke^{-r\tau}$:

$$\partial P/\partial r = \partial C/\partial r - K\tau e^{-r\tau} = -K\tau e^{-r\tau}[1 - \Phi(d_2)] = -K\tau e^{-r\tau}\Phi(-d_2) < 0.$$

[**3**]

(iv) *Financial interpretation.*

As above: as r increases, stock *sellers* also operate in a buyer's market, but this is against them. So for *puts* (options to sell), $\rho < 0$. **[3]**

(v) *American options.*

All this extends to American options,via the *Snell envelope*, which is *order-preserving*. The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As r increases, the Z-terms increase for calls (rho is positive for European calls). As the Zs increase, the Us increase (above: backward induction on n – dynamic programming, as usual for American options). Combining: as r increases, the U-terms increase. So rho is also positive for American calls. Similarly, rho is negative for American puts. **[4]**

[Similar to 'vega positive', done in Problems]

Q2. *Forward-rate agreements (FRAs)*

A *forward-rate agreement (FRA)* is a contract involving *three* times: the *current time* t ('now'), the *expiry time* $T > t$, and the *maturity time* $S > T$. The contract gives the holder an interest-rate payment for the period from T to S with fixed rate K at maturity S against an interest-rate payment over the same period with rate $L(T, S)$. So this contract allows the holder to lock in the interest rate between T and S at a desired value K. **[3]**

The FRA is called a *receiver FRA* if we pay floating $L(T, S)$ (floating: uncertain, and in the future) and receive fixed K. It is a *payer FRA* if we pay K and receive floating $L(T, S)$. **[2]**

Proposition. The price of a receiver FRA is

$$FRA(t, T, S, K) = P(t, S)(S - T)K - P(t, T) + P(t, S). \qquad (FRA)$$

The price of a payer FRA is the negative of this.

Proof. The second statement follows from the first, as the cash flows for receivers and payments go in opposite directions. Write $\tau := S - T$. As payments are made at S, we need to discount them back to t through $D(t, S)$:

$$\begin{aligned} FRA(t, T, S, K) &= E_t[D(t, S)\tau K - D(t, S)\tau L(T, S)] \quad \text{(def. of rec. FRA)} \\ &= \tau K E_t[D(t, S)] - E_t[D(t, S)\tau L(T, S)] \\ &= \tau K P(t, S) - E_t[D(t, S)\tau L(T, S)] \quad \text{(by } (P - D)) \\ &= \tau K P(t, S) - E_t[\tau D(t, T)D(T, S)L(T, S)] \quad \text{(definition of } D) \end{aligned}$$

$$= \tau K P(t, S) - E_t[E_T[\tau D(t, T)D(T, S)L(T, S)]] \quad \text{(tower property).} \qquad \mathbf{[3]}$$

Now $L(T, S) = (1 - P(T, S))/(\tau P(T, S))$ and $P(T, S) = E_T[D(T, S)]$ is $\mathcal{F}_T$-measurable (= known at time T). **[2]**

So (taking out what is known)

$$E_t[E_T[\tau D(t, T)D(T, S)L(T, S)]] = E_t[\tau D(t, T)L(T, S)E_T[D(T, S)]]$$

$$= E_t[\tau D(t, T)L(T, S)P(T, S)] \quad \text{(definition of } P(T, S))$$

$$= E_t[D(t, T)] - E_t[D(t, T)P(T, S)] \quad \text{(definition of } L(T, S))$$

$$= E_t[D(t, T)] - E_t[D(t, T)E_T[D(T, S)]] \quad \text{(definition of } P(T, S))$$

$$= E_t[D(t, T)] - E_t[E_T[D(t, T)D(T, S)]]] \quad \text{(putting } E_t,\ E_T \text{ together)}$$

$$
\begin{aligned}
&= E_t[D(t,T)] - E_t[D(t,T)D(T,S)]] \quad \text{(tower property)} \\
&= E_t[D(t,T)] - E_t[D(t,S)]] \quad \text{(definition of } D(.,.)) \\
&= P(t,T) - P(t,S) \quad \text{(definition of } P(.,.)).
\end{aligned}
$$

Combining,

$$FRA(t,T,S,K) = \tau K P(t,S) - P(t,T) + P(t,S). \qquad // \qquad [\mathbf{10}]$$

[Seen – lectures]

Q3 *Ornstein-Uhlenbeck (OU)/Vasicek (Vas) process.*
(i) The OU SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[2]**
(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta Ce^{-\beta t}dt + e^{-\beta t}dC = -\beta V dt + e^{-\beta t}dC,$$

so V is a solution of (OU) if $e^{-\beta t}dC = \sigma dW$, $dC = \sigma e^{\beta t}dW$, $C = c + \sigma \int_0^t e^{\beta u}dW$. So with initial velocity v_0, $V = e^{-\beta t}C$ is

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t}\int_0^t e^{\beta u}dW_u. \qquad \textbf{[3]}$$

(iii) V comes from W, Gaussian, by linear operations, so is Gaussian.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u}dW_u] = \int_0^t e^{\beta u}E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t}\int_0^t e^{\beta u}dW_u)^2] = \sigma^2 e^{-2\beta t}\int_0^t (e^{\beta u})^2 du$$

$$= \sigma^2 e^{-2\beta t}[e^{2\beta t} - 1]/(2\beta) = \sigma^2[1 - e^{-2\beta t}]/(2\beta).$$

So V_t has distribution $N(v_0 e^{-\beta t}, \sigma^2(1-e^{-2\beta t})/(2\beta))$. (So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$, the *Maxwell-Boltzmann distribution* of Statistical Mechanics.) **[3]**
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which is

$$\sigma^2 E[e^{-\beta t}\int_0^t e^{\beta v}dW_v . e^{-\beta(t+u)}(\int_0^t + \int_t^{t+u})e^{\beta w}dW_w].$$

By independence of Brownian increments, $\int_t^{t+u}$ contributes 0, so by above

$$cov(V_t, V_{t+u}) = e^{-\beta u}var(V_t) = \sigma^2 e^{-\beta u}[1-e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty).$$

[3]
(v) V is Markov (a diffusion), being the solution of the SDE (OU). **[2]**
(vi) The process shows *mean reversion* – a strong push towards the central value. This is characteristic of interest rates (under normal conditions). The financial relevance is to the *Vasicek model* of interest-rate theory. **[3]**
(vii) The Vasicek model is widely used because it is analytically tractable,

and easy to interpret. Its main drawbacks both stem from its Gaussianity (as do its main advantages!):
(a) negative interest rates;
(b) poor fit to market data: tails too thin, symmetric rather than skew, etc.
In addition:
(c) One-factor models are not capable of capturing all relevant aspects; one needs at least a two- (or three-) factor model, and the Vasicek model does indeed extend easily to higher factors. **[4]**
[Seen, lectures.]

Q4. *Breeden-Litzenberger formula*

By *Black's caplet formula*,

$$Cpl(0, T_1, T_2, K) = P(0, T_2)\tau[F_2(0)\Phi(d_1) - K\Phi(d_2)],$$

$$d_1, d_2 := \frac{\log(F_2(0)/K) \pm \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1}v_1(T_1)}:$$

$$Cpl(0, T_1, T_2, K) = P(0, T_2)\tau Bl(K, F_2(0), v_2(T_1)),$$

say. **[2]**

Let p_2 be the density of $F_2(T_1)$ under the T_2-forward measure (if Black's formula were exact, this density would be lognormal). As caplets are options,

$$\begin{aligned} Cpl(0, T_1, T_2, K) &= P(0, T_2)\tau Bl(K, F_2(0), v_2(T_1)) \\ &= P(0, T_2)\tau E_0^2[(F(T_1, T_1, T_2) - K)_+] \\ &= P(0, T_2)\tau \int (x - K)_+ p_2(x)dx. \qquad (*) \end{aligned}$$ **[5]**

Theorem (Breeden-Litzenberger formula (1978)). With enough smoothness to differentiate $(*)$ under the integral sign, the density $p_2(K)$ is given by the *second partial derivative of the caplet price w.r.t. the strike* K, via

$$p_2(K).\tau P(0, T_2) = \frac{\partial^2}{\partial K^2} Cpl(0, T_1, T_2, K). \qquad (BL)$$ **[3]**

Proof. Differentiating $(*)$: as

$$(\partial/\partial K)[(x - K)_+] = -I(K < x)$$

(the derivative does not exist at the point x, but as this point contributes nothing to $\int ...dx$ this makes no difference), this gives

$$\begin{aligned} \frac{\partial}{\partial K} Cpl(0, T_1, T_2, K) &= P(0, T_2)\tau \int -I(K < x)p_2(x)dx \\ &= -P(0, T_2)\tau \int_K^\infty p_2(x)dx. \end{aligned}$$

Differentiating both sides gives (BL). **[7]**

To use this in practice, one would need data for (at least) three nearby strikes, K, $K \pm \Delta K$, say, and would use a second-order finite-difference approximation

$$\frac{\partial^2}{\partial K^2} C \sim [C(K + \Delta K) - 2C(K) + C(K - \Delta K)]/(\Delta K)^2$$

(in an obvious notation). **[4]**
[Seen – lectures]

Q5. *Corporate bonds; junk bonds; credit rating; credit scoring*
Corporate bonds

The bond market splits, into two: *government bonds* ('gilts', in UK), regarded as free of default risk with major developed countries (UK, US etc.), though not for, say, third-world countries, and *corporate bonds*. Here the money is to be lent to a *company* (corporation, US), and companies can default – companies can go bankrupt, and disappear (countries cannot ...). So investors considering buying a corporate bond will, naturally, want to know as much as possible about the company's *credit-worthiness*.

In the past, it was only big established firms that could raise capital by issuing their own bonds. Smaller or less-established firms thereby felt excluded (unable to raise capital, and so unable to expand, re-equip etc.) This unsatisfied demand created a potential market. 'Nature abhors a vacuum', and this market was eventually created. **[5]**

Junk bonds

This term was introduced in the USA in the 1980s, for high-risk, high-yield bonds issued by companies that would not previously have been able to raise capital in this way (see above). The driving force behind them was the "junk-bond king", the financier Michael Milken (1946-). Convicted in 1989 and serving two years in prison, he was the inspiration behind the part played by Michael Douglas in the 1987 film *Wall Street*, Gordon Gekko (a suitably reptilian name – a gekko is a kind of lizard).

Junk bonds were regarded as a boon by companies newly enabled to issue them (and Milken as a hero). They were much used as securities used to leverage (finance on borrowed money) hostile takeovers ('buyouts', US) in the 1980s. The term is used nowadays for bonds rated BB or lower. **[5]**

Credit rating

Investors find it convenient to have some 'third-party' assessment of the credit-worthiness of a firm whose bonds they are considering purchasing. These are typically given in letter-grade terms modelled on student grades: AAA (triple A), AA, A, B, BB, ..., and the analogy with student grades is apt. With classified degrees, as in the UK now, the university puts its reputation behind the degree class. This spares an employer having to use time (scarce) and judgement (of the perhaps too unfamiliar) to form his own assessment, based on applicant-supplied information.

A number of firms began to specialise in providing such ratings; this role was recognised in 1975 by the SEC (Securities and Exchange Commission). The main three today are Standard & Poor's, Moody's and Fitch.

In the case of Lehman, the ratings agencies did not pick up the dramatic slide of Lehman towards insolvency. Their credit ratings for Lehman, which remained good, increasingly departed from reality.

This raises the question of *conflict of interest.* Firms *pay* the agencies to carry out their credit rating. The agency thus acquires an interest in keeping the firm's business by giving or keeping a good grade. **[5]**

Credit scoring

Banks make their money by lending to customers (individual or corporate) at a higher rate of interest than they pay to their customers' savings accounts. When a firm, or a person, asks for a loan, the bank will need to assess their credit-worthiness. There may well be an established relationship with the firm or person, in which case their track record is there for the bank, and will be the main basis for judgement. Without such a track record (and, for safety's sake, even with), the bank will ask the applicant to supply a lot of background information to help it make its decision. Thus a person or couple might be asked for such things as (covariates of loan-worthiness):
age; marital/family status; health record; employment status and record; income; home-ownership status (and value of home if owned), etc.

The relevant areas of Statistics concerned with such covariates include Regression, the Linear Model, Generalised Linear Models (GLMs), Survival Analysis and the Cox *proportional hazards model* in Survival Analysis, as applied to life insurance. The analogy between death and default is clear, and so proportional hazards became widely used by banks in assigning *credit scores* to loan applicants. **[5]**

[Seen – lectures]