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II. INTEREST-RATE PRODUCTS AND DERIVATIVES

1. Terminology

Numéraire

Recall (MATL480) that a *numéraire* (or just numeraire, dropping the accent for convenience) is any asset whose value at any time t is always-positive, and so can be used as a *unit of reckoning.* As we discussed there, one can change numeraire (indeed, one needs to when switching from one currency to another), but this does not alter anything of importance. So we may choose our numeraire to suit our convenience. We shall usually use the *bank account* as numeraire.

Bank account

We shall use as our bank account – numeraire of choice – the always-positive process with initial value $B_0 > 0$ and dynamics

$$dB_t = r_t B_t dt, \qquad B_t = B_0 \exp(\int_0^t r_s ds).$$

Here $r = (r_t)$ gives the risk-free instantaneous interest rate – *spot rate*, or *short rate* ('short' for short-term) – at time t.

Risk-Neutral Valuation Formula (RNVF)

Recall the RNVF, and write $\mathbb{Q} = \mathbb{Q}^B$ for the risk-neutral measure (EMM) denominated in numeraire B, with expectation which we may abbreviate to $E^{\mathbb{Q}}$ or E^B. Then using E_t as shorthand for $E[.|\mathcal{F}_t]$ – conditional expectation given what we know at time t – we have, for an asset $H = (H_t)$ (which we think of as the *payoff* – at time T) the time-t value

$$E_t^B[\frac{B_t}{B_T} H_T] = E_t^{\mathbb{Q}}[\exp(-\int_t^T r_s ds) H_T]. \qquad (RNVF)$$

As above, we can (and will) abbreviate notation here to

$$E_t[\frac{B_t}{B_T} H_T] = E_t[\exp(-\int_t^T r_s ds) H_T]. \qquad (RNVF)$$

Recall also that in the 'risk-neutral world' – i.e., after the Girsanov change of measure $\mathbb{P} \mapsto \mathbb{Q}$ – assets have as drift the risk-free interest rate r: for an asset $A = (A_t)$, with volatility $Avol$,

$$dA_t = r_t A_t dt + Avol_t dBM_t^{\mathbb{Q}},$$

with $BM^{\mathbb{Q}}$ denoting Brownian motion (BM) under the measure $\mathbb{Q}$.

Zero-Coupon Bonds (ZCB) – "bonds" – and their prices

Coupons are payments made during a loan period, in addition to repayment of the principal (the amount borrowed) and interest at the end. They correspond to *dividends* paid to shareholders. When money is lent without coupons: a *T-maturity zero-coupon bond (ZCB)* is a contract which guarantees the payment of one unit of currency at time T. The contract value at time $t \in [0, T]$ is denoted by $P(t, T)$:

$$P(T, T) = 1,$$

and writing $D = D(t, T)$ for the *discount function*

$$D(t, T) := \exp(-\int_t^T r_s ds), \tag{D}$$

the price (value) at t is

$$P(t, T) = E_t[\frac{B_t}{B_T} 1] = E_t[\exp(-\int_t^T r_s ds)] = E_t[D(t, T)]. \tag{P}$$

ZCBs can be used as the fundamental quantities – the building bricks from which other more complicated things can be derived.

We will drop explicit mention of $\mathbb{Q}$ in $E_t^{\mathbb{Q}}[.]$ from now on: all our expectations will be under $\mathbb{Q}$ unless otherwise stated.

We define the *continuously-compounded spot interest rate* at time t for maturity at time T as $R(t, T)$, where R is defined by

$$P(t, T) = \exp\{-R(t, T)(T - t)\}. \tag{R}$$

We can alternatively write

$$P(t, T) = \exp\{-\int_t^T f(t, u) du\},$$

where

$$f(t,T) := -\frac{\partial}{\partial T}\log P(t,T) \qquad (f)$$

is called the *instantaneous forward rate.* We return to it in Ch. IV, as an alternative to modelling the spot-rate r_t.

LIBOR

The *spot-LIBOR rate* $L(t,T)$ at time t for maturity T is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when accruing occurs *proportionally* to the investment time:

$$P(t,T)(1+(T-t)L(t,T)) = 1, \qquad L(t,T) = \frac{1-P(t,T)}{(T-t)P(t,T)}. \qquad (P-L)$$

This (the definition of LIBOR!) is an *extremely important relationship!* For, when one of P and L goes *up*, the other goes *down* – the two are *inversely related.* We shall see below that this is the key to *quantitative easing (QE).*

Quantitative easing (QE)

From the Bank of England's website (2018):

"Home / Monetary policy / Quantitative easing
Quantitative easing
The Bank of England can purchase assets to stimulate the economy. This is known as quantitative easing. In this section:
Inflation
The interest rate (Bank Rate)
On this page:
What is quantitative easing?

Quantitative easing is an 'unconventional' form of monetary policy that our Monetary Policy Committee has carried out in order to stimulate the economy when interest rates are already low. The ultimate aim of this is to boost spending to reach our inflation target of 2 %. Quantitative easing is sometimes called 'QE' or just 'asset purchases'. The Monetary Policy Committee makes decisions on quantitative easing at the same time as it makes its interest-rate decision.

Quantitative easing does not involve literally printing more money. Instead, we create new money digitally.

Quantitative Easing Asset Purchase Programme:
£435bn
Next due: 8 February 2018
Corporate Bond purchases:
£10bn
Next due: 8 February 2018
What is quantitative easing?

Quantitative easing is when a central bank like the Bank of England creates new money electronically to make large purchases of assets. We make these purchases from the private sector, for example from pension funds, high-street banks and non-financial firms. Most of these assets are government bonds (also known as gilts). The market for government bonds is large, so we can buy large quantities of them fairly quickly.

The purchases are of such a scale that they push up the price of assets, lowering the yields (the return) on them[1]. This encourages those selling these assets to us to use the money they received from the sale to buy assets with a higher yield instead, like company shares and bonds.

As more of these other assets are bought, their prices rise because of the increased demand. This pushes down on yields in general. The companies that have issued these bonds or shares benefit from cheaper borrowing because of these lower yields, encouraging them to spend and invest more."

Read, mark, learn and inwardly digest this clear statement of a vitally important public policy. This in a nutshell is one of the two most important things in the course (the other being market models – Part II).

The QE programmes, applied extensively after the Crash of 2007/8, have been broadly successful in keeping interest rates low (LIBOR here acts as a proxy for interest rates generally), so as to stimulate the economy, from its zombie-like existence post-Crash towards more normal levels of activity as per pre-Crash.

Effects of QE.

In accordance with the Law of Unintended Consequences, QE has had extensive, unanticipated and undesirable side-effects. For, QE discourages investors from buying *bonds* (because the yields are low), and encourages them to buy *stock* instead. This increases stock prices, and so has resulted

[1] by $(P - L)$

in record highs for the FTSE, although the economy generally remains depressed. Now these purchases are made by the already well-off, who have benefited financially; the economy remaining depressed means that these benefits are not shared by those without the money to buy such assets. This has gravely reduced *social mobility*, and generated great resentment by those who feel left out, though no fault of their own, while seeing those with money using that money to make more money. This is the *Matthew Principle*: 'For unto him that hath, it shall be given, but from him that hath not it shall be taken away, even that which he hath'. The resulting political tensions have diminished social cohesion, poisoned politics, and underlie recent events such as Brexit.

The *zero-coupon curve* – often called the *yield curve*, or *term structure* at time t (present time, now) – is the graph of the function

$$T \mapsto L(t,T), \quad \text{initial point} \;\; r_t = \lim_{T \downarrow t} L(t,T) \sim L(t, t+\epsilon).$$

This function is called the *term structure of interest rates* at time t. So the fundamental quantity here is the process $r = (r_t)$, from which $P(t,T)$ is obtained by an expectation, $E_t = E_t^{\mathbb{Q}}$.

Matters split according to whether a financial product depends on the *curve dynamics* – the way in which the yield curve changes over time – or not. As there are many different models for the yield-curve dynamics, we split our treatment of products accordingly.

In addition to the term structure of interest rates (above), we consider the *term structure of volatility* (V.13 below: 'smile dynamics').

2. Products not depending on the curve dynamics

Forward-rate agreements (FRAs)

A *forward-rate agreement (FRA)* is a contract involving *three* times: the *current time* t ('now'), the *expiry time* $T > t$, and the *maturity time* $S > T$. The contract gives the holder an interest-rate payment for the period from T to S with fixed rate K at maturity S against an interest-rate payment over the same period with rate $L(T,S)$. So this contract allows the holder to lock in the interest rate between T and S at a desired value K.

The FRA is called a *receiver FRA* if we pay floating $L(T,S)$ (floating: uncertain, and in the future) and receive fixed K. It is a *payer FRA* if we

pay K and receive floating $L(T,S)$. To remember which is which: it is K that decides (K is analogous to the strike price for an option (MATL480); hence this choice of notation).
Note. To remember this: it is K that is *known* at the time the agreement is made (as with the strike K for call and put options); $L(T,S)$, which is floating, is *unknown*. So, "pay or receive K" gives the names to *payer* or *receiver* FRAs.

Proposition. The price of a receiver FRA is

$$FRA(t,T,S,K) = P(t,S)(S-T)K - P(t,T) + P(t,S). \qquad (FRA)$$

The price of a payer FRA is the negative of this.

Proof. The second statement follows from the first, as the cash flows for receivers and payments go in opposite directions. Write $\tau := S - T$. As payments are made at S, we need to discount them back to t through $D(t,S)$:

$$\begin{aligned} FRA(t,T,S,K) &= E_t[D(t,S)\tau K - D(t,S)\tau L(T,S)] \quad \text{(def. of rec. FRA)} \\ &= \tau K E_t[D(t,S)] - E_t[D(t,S)\tau L(T,S)] \\ &= \tau K P(t,S) - E_t[D(t,S)\tau L(T,S)] \quad \text{(by } (P-D)) \\ &= \tau K P(t,S) - E_t[\tau D(t,T)D(T,S)L(T,S)] \quad \text{(definition of } D) \\ &= \tau K P(t,S) - E_t[E_T[\tau D(t,T)D(T,S)L(T,S)]] \quad \text{(tower property).} \end{aligned}$$

Now $L(T,S) = (1-P(T,S))/(\tau P(T,S))$ and $P(T,S) = E_T[D(T,S)]$ is $\mathcal{F}_T$-measurable (= known at time T). So

$$E_t[E_T[\tau D(t,T)D(T,S)L(T,S)]] = E_t[\tau D(t,T)L(T,S)E_T[D(T,S)]]$$

(taking out what is known)

$$= E_t[\tau D(t,T)L(T,S)P(T,S)] \quad \text{(definition of } P(T,S))$$

$$= E_t[D(t,T)] - E_t[D(t,T)P(T,S)] \quad \text{(definition of } L(T,S))$$

$$= E_t[D(t,T)] - E_t[D(t,T)E_T[D(T,S)]] \quad \text{(definition of } P(T,S))$$

$$= E_t[D(t,T)] - E_t[E_T[D(t,T)D(T,S)]]] \quad \text{(putting } E_t,\ E_T \text{ together)}$$

$$= E_t[D(t,T)] - E_t[D(t,T)D(T,S)]] \quad \text{(tower property)}$$

$$= E_t[D(t,T)] - E_t[D(t,S)]] \quad \text{(definition of } D(.,.))$$
$$= P(t,T) - P(t,S) \quad \text{(definition of } P(.,.)).$$

Combining,

$$FRA(t,T,S,K) = \tau K P(t,S) - P(t,T) + P(t,S). \qquad //$$

Forward LIBOR rate

The value of K which makes the contract fair (both sides 0) is the *forward LIBOR interest rate* at time t for expiry T and maturity S,

$$K = F(t;T,S)$$

say. This is obtained by solving for K in LHS = RHS = 0 above:

$$\tau K P(t,S) - P(t,T) + P(t,S) = 0:$$

$$K = F(t;T,S) := \frac{1}{(S-T)}\Big(\frac{P(t,T)}{P(t,S)} - 1\Big) = \frac{1}{(S-T)}(P(t,T)/P(t,S) - 1).$$

Comparing RHSs (the third in the first string of equations with the last in the second) gives

$$\tau E_t[D(t,S)L(T,S)] = P(t,T) - P(t,S),$$

which (recall $\tau := S - T$) with the definition of F above gives:

Corollary

$$E_t[D(t,S)L(T,S)] = P(t,S)F(t;T,S). \qquad (*)$$

Note that the derivation above is very general. We have made no assumptions – so all this holds for any model of interest rates. All we have used is the definitions of the quantities involved and standard measure theory (MATL480 – tower property, taking out what is known, etc.).

Interest Rate Swaps (IRS)

An *interest-rate swap (IRS)* is a contract that exchanges payments between two differently indexed legs, starting from a future time-instant. At future dates $T_i, \cdots, T_k$, writing $\tau_j := T_j - T_{j-1}$ (following the notation used above), the *fixed leg* pays the *floating leg*

$$\tau_j K \qquad \text{(fixed to floating)}$$

and the floating leg pays the fixed leg

$$\tau_j L(T_{j-1}, T_j),$$

(the LIBOR rate), or taking the expectation $E_{T_i}[.]$,

$$\tau_j F(t_i; T_{j-1}, T_j).$$

Here we have two different interest rates (in two different countries, say); we take the view of the interest rate in our country/the country where we are operating. Then, K is fixed; the IRS is called a *payer IRS* for the company *paying* K above and *receiver IRS* for the company *receiving* K.

The *discounted payoff* at time $t < T_i$ of a receiver IRS is

$$\sum_{j=i+1}^{k} D(t, T_i)\tau_j (K - L(T_{j-1}, T_j)).$$

Interest-Rate Swaps and Forward-Rate Agreements

Note that a IRS can be broken down into, and so can be valued (or priced) as, a collection of FRAs. In particular, a receiver IRS (RIRS for short) can be valued as a collection of receiver FRAs (RFRAs):

$$RIRS(t, [T_i, \cdots, T_k], K) = \sum_{j=i+1}^{k} RFRA(t, T_{j-1}, T_j, K),$$

which by the Proposition is

$$\sum_{j=i+1}^{k} \tau_j K P(t, T_j) - P(t, T_i) + P(t, T_k) \qquad (a)$$

(the sums of the last two terms in (FRA) 'telescope' – cancel, leaving just two end-terms), or alternatively

$$\sum_{j=i+1}^{k} \tau_j (K - F(t; T_{j-1}, T_j)).$$

Similarly for payer IRSs and payer FRAs.

The value

$$K = S_{i,k}(t)$$

which makes

$$IRS(t, [T_i, \cdots, T_k], K) = 0$$

(payer or receiver here) is called the *forward swap rate*. We abbreviate:

$$F_j(t) := F(t, T_{j-1}, T_j).$$

Solving for K, the forward swap rate is

$$S_{i,k}(t) = \frac{P(t, T_i) - P(t, T_k)}{\sum_{j=i+1}^{k} \tau_j P(t, T_j)} : \qquad (b)$$

$$S_{i,k}(t) = \sum_{j=i+1}^{k} w_j(t) F_j(t), \qquad w_j(t) := \frac{\tau_j P(t, T_j)}{\sum_{j=i+1}^{k} \tau_j P(t, T_j)}.$$

Here the weights $w_j(t)$ (which are functions of the Fs, and so depend on the future at time t, i.e. are random at t) are non-negative and sum to 1:

$$0 \leq w_j(t) \leq 1, \sum_{j=i+1}^{k} w_j(t) = 1.$$

Combining (a) and (b),

$$RIRS(t, [T_i, \cdots, T_k], K) = (K - S_{i,k}(t)) \sum_{j=i+1}^{k} \tau_j P(t, T_j),$$

and likewise (reversing the cash flows)

$$PIRS(t, [T_i, \cdots, T_k], K) = (S_{i,k}(t) - K) \sum_{j=i+1}^{k} \tau_j P(t, T_j).$$

Note. We will encounter summations like those here many times. They illustrate our basic method: break down the (uncountable) time continuum into finitely many intervals; each contributes a summand to the summation. This *decomposition* reduces an infinite-dimensional problem to a finite-dimensional one, which is *much easier*. A good example of this is the reduction of caps to caplets (and floors to floorlets); see II.3 below.

3. Products depending on the curve dynamics

Caps and caplets

A *cap* can be viewed as a payer IRS where each exchange payment is made only if if has positive value (so this is like an *option*, which will only be exercised if it pays the holder to do so!). So the cap discounted payoff is

$$\sum_{j=i+1}^{k} D(t,T_j)\tau_j(L(T_{j-1},T_j)-K)_+ = \sum_{j=i+1}^{k} D(t,T_j)\tau_j(F_j(T_{j-1})-K)_+.$$

Suppose a company is LIBOR-indebted, and has to pay at times $T_{i+1},\cdots,T_k$ the LIBOR rates resetting at the previous time-instants $T_i,\cdots,T_{k-1}$. The company thinks that LIBOR rates will increase in the future (so its debt will increase), and wishes to protect itself. It can buy a cap, so that it pays at most K at each payment date (so K is like the strike price for an option).

A cap contract can be decomposed into a sum of terms, the summands in the above formula. These are called *caplets*:

$$D(t,T_j)\tau_j(L(T_{j-1},T_j)-K)_+ = D(t,T_j)\tau_j(F_j(T_{j-1})-K)_+.$$

Each caplet can be evaluated separately, and the values summed to obtain the cap price (note the analogy with call options!). But to value caplets, we need the whole distribution of future rates, not just their means. So the dynamics of interest rates is needed to value caplets: the current zero curve $T\mapsto L(t,T)$ is not enough. We need to specify how this infinite-dimensional object moves, in order to find its distribution in the future. This can be done, for example, by specifying how the spot-rate r moves (spot-rate models: Ch. III below).

We will price caplets by *Black's caplet formula* (V.2; 4a). This is a close relative of the *Black-Scholes formula* of MATL480.

Floors and floorlets

A *floor* can be seen as a receiver IRS where each exchange payment is made only if it has positive value (as with caps above, and again, as with options – but this time, with puts rather than calls). The floor discounted payoff is

$$\sum_{j=i+1}^{k} D(t,T_j)\tau_j(K-L(T_{j-1},T_j))_+ = \sum_{j=i+1}^{k} D(t,T_j)\tau_j(K-F_j(T_{j-1}))_+.$$

The floor price is the risk-neutral expectation $E = E^{\mathbb{Q}}$ of the above discounted payoff. The *floorlets* are the individual summands above.

Swaptions

Finally, we introduce options on interest-rate swaps (IRSs) – *swaptions.*

A payer swaption is a contract giving the right (but not – understood from now on – the obligation) to enter into a payer IRS at some future time. This time of possible entry is called the *maturity.* Usually, the maturity is the first reset time of the underlying IRS.

The IRS value at its first reset date T_i, i.e. at maturity, is by above

$$\begin{aligned} PIRS(t, [T_i, \cdots, T_k], K) &= \sum_{j=i+1}^{k} P(t_i, T_j)\tau_j(F(T_i; T_{j-1}, T_j) - K) \\ &= (S_{i,k}(t) - K) \sum_{j=i+1}^{k} \tau_j P(t, T_j) \\ &= (S_{i,k}(t) - K)C_{i,k}(T_i), \end{aligned}$$

say. The option will be exercised only if this is positive. So the payer-swaption's discounted payoff at time t is

$$D(t, T_i)C_{i,k}(T_i)(S_{i,k}(T_i)-K)_+ = D(t, T_i)\Big(\sum_{j=i+1}^{k} P(t_i, T_j)\tau_j(F(T_i; T_{j-1}, T_j)-K)\Big)_+.$$

Unlike caps, this payoff *cannot be decomposed additively.* To summarise:

Caps:

Decompose into caplets, each with a single forward rate. Deal with each caplet separately, and add the results together. Only *marginal* distributions of the different forward rates are involved.

Swaptions.

No such decomposition is possible. The summation is *inside* the positive-part operator $(.)_+$, not outside as with caps. With swaptions, we need to consider the *joint* distributions involved. The *correlation* between rates is fundamental in handling swaptions, unlike the cap case.

Which variables do we model?

For some products – FRAs, IRSs – the *dynamics* of interest rates are not needed for valuation. The *current curve* is enough. But for caps, swaptions and more complex derivatives, the dynamics *are* necessary. Specifying a stochastic dynamics for interest rates amounts to choosing an *interest-rate model*. There are many!

Which quantities do we model? Short rates r_t? LIBOR rates $L(t,T)$? Forward LIBOR rates $F_j(t) = F(t; T_{j-1}, T_j)$? Forward swap rates $S_{i,k}(t)$? Bond prices $P(t,T)$?

How is randomness modelled? – i.e., what kind of stochastic process or SDE do we select for our model? (Markov diffusions, etc.)

What are the consequences of our choice in terms of valuation of market products, ease of implementation, goodness of calibration to real data, pricing complicated products with the calibrated model, possibilities for diagnostics on the model outputs and implications, stability, robustness etc?

The above shows that this is a large and complex field. But, although the field is interesting and challenging academically, it is by no means only – or even primarily – of academic interest. The market in interest-rate derivatives is vast – trillions of dollars. A very small percentage of such a vast sum is still very large. Interest rates and interest-rate derivatives are accordingly reckoned, not in percentages but in *basis points (bp)* – hundredths of a percent. A gain or loss of a few bp is crucial in the financial services industry in general and in money markets in particular. This is why we study the subject in such detail!

There are many interest-rate products on the market, but their number is measured in hundreds rather than thousands (Government bonds, for example, are auctioned at fixed times, at each of which only a fixed range of bond maturities is offered for sale). Because of the vast amounts of money involved (trillions, as above), each product is traded in great numbers and amounts. This means that interest-rate markets are *highly liquid* – there is a great deal of trading, and so prices are known accurately.

Other products. We mention here:
Inflation-linked products. European governments have been issuing inflation-linked bonds since the early 1980s, but it is only recently that such products have become more commonly traded. Inflation here is measured in terms of the Consumer Price Index (CPI). Various measures of inflation are used, e.g. with or without housing costs included. The CPI is based on a representative basket of goods and services. For details, see [BM, Part VI, Ch. 15 - 20].
Ratchets. See e.g. [BM, 13.5].
Constant maturity swaps (CMS). See e.g. [BM, 13.7].