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3. G2++: Gaussian two-factor additive models

We follow [BM, 4.2]. Taking the mean reversion level to be 0 for convenience, we recall the model:

$$r_t = x_t + y_t + \phi(t), \qquad r(0) = r_0, \tag{G2++}$$

where

$$dx_t = -ax_t dt + \sigma dW_1(t), \qquad x(0) = x_0,$$
$$dy_t = -by_t dt + \eta dW_2(t), \qquad y(0) = y_0,$$

where (W_1, W_2) is a two-dimensional correlated Brownian motion with correlation ρ:

$$dW_1(t)dW_2(t) = \rho dt.$$

Here r_0, a, b, σ, η are positive constant, and $\rho \in [-1, 1]$. The function ϕ is deterministic, and defined in the time-interval $[0, T^*]$, where T^* is the relevant time-horizon, typically 10 years, 30y or 50y. In particular,

$$\phi(0) = r_0$$

(to get a fit at the initial time $t = 0$). We write $\mathcal{F}_t$ for the σ-field generated by the bivariate process (x, y) up to time t.

The SDE above is of OU type. Proceeding as there, we find, for each $s \in [0, t]$

$$\begin{aligned} r_t &= x_s e^{-a(t-s)} + y + s e^{-b(t-s)} \\ &\quad + \sigma \int_s^t e^{-a(t-u)} dW_1(u) \\ &\quad + \eta \int_s^t e^{-b(t-u)} dW_2(u) + \phi(0). \end{aligned}$$

So $r_t | \mathcal{F}_s$ is normal, with conditional mean and variance

$$E[r_t | \mathcal{F}_s] = x_s e^{-a(t-s)} + y_s e^{-b(t-s)} + \phi(t),$$

$$var(r_t | \mathcal{F}_s) = \frac{\sigma^2}{2a}[1 - e^{-2a(t-s)}] + \frac{\eta^2}{2b}[1 - e^{-2b(t-s)}] + 2\rho \frac{\sigma\eta}{a+b}[1 - e^{-(a+b)(t-s)}].$$

In particular,

$$r_t = \sigma \int_s^t e^{-a(t-u)} dW_1(u) + \eta \int_s^t e^{-b(t-u)} dW_2(u) + \phi(0).$$

Bond pricing

As before,

$$P(t,T) = E_t[\exp\{-\int_t^T r_s ds\}],$$

with $P(t,T)$ the price at time t of a ZCB with unit face value (payoff 1) maturing at T, and $E = E_{\mathbb{Q}}$. To compute this, we need the following Lemma (the proof is not difficult, but is omitted).

Lemma. In the above, the random variable

$$I(t,T) := \int_t^T [x_u + y_u] du$$

conditional on $\mathcal{F}_t$ is normal with mean

$$M(t,T) = \frac{1 - e^{-a(T-t)}}{a} x_t + \frac{1 - e^{-b(T-t)}}{b} y_t$$

and variance

$$\begin{aligned} V(t,T) &= \frac{\sigma^2}{a^2}[T - t + \frac{2}{a}e^{-a(T-t)} - \frac{1}{2a}e^{-2a(T-t)} - \frac{3}{2a}] \\ &+ \frac{\eta^2}{b^2}[T - t + \frac{2}{b}e^{-b(T-t)} - \frac{1}{2b}e^{-2b(T-t)} - \frac{3}{2b}] \\ &2 + \rho\frac{\sigma\eta}{ab}[T - t + \frac{e^{-a(T-t)} - 1}{a} + \frac{e^{-b(T-t)} - 1}{b} - \frac{e^{-(a+b)(T-t)} - 1}{a+b}]. \end{aligned}$$

Proposition. In the above,

$$P(t,T) = \exp\{\int_t^T \phi(u) du - M(t,T) + \frac{1}{2}V(t,T)\}.$$

Proof. As ϕ is deterministic, this follows from the Lemma and the fact that if $Z \sim N(m_Z, \sigma_Z^2)$, then (taking $t = 1$ for the argument of the MGF)

$$E[\exp\{Z\}] = \exp\{m_Z + \frac{1}{2}\sigma_Z^2\}. \qquad //$$

Now suppose that the term structure of discount factors $D(t,T)$ that is currently observed in the market (recall $P(t,T) = E[D(t,T)]$) is given by the sufficiently smooth function

$$T \mapsto P^M(0,T)$$

('M for market' here). If $f^M(0,T)$ is the corresponding instantaneous forward rate at time 0 for maturity T, i.e.

$$f^M(0,T) = \frac{\partial}{\partial T} \log P^M(0,T),$$

then we have the following (proof omitted: [BM, 4.2.2]):

Proposition. The G2++ model fits the currently-observed term structure of discount factors iff, for each T,

$$\begin{aligned} \phi(T) &= f^M(0,T) + \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 + \frac{\eta^2}{2b^2}(1-e^{-bT})^2 \\ &\quad + \rho\frac{\sigma\eta}{ab}(1-e^{-aT})(1-e^{-bT}), \end{aligned}$$

i.e. iff

$$\exp\{-\int_t^T \phi(u)du\} = \frac{P^M(0,T)}{P^M(0,t)} \exp\{-\frac{1}{2}[V(0,T) - V(0,t)]\},$$

so that the corresponding ZCB prices at t are given by

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\{\mathcal{A}(t,T)\},$$

where

$$\mathcal{A}(t,T) := \frac{1}{2}[V(t,T) - V(0,T) + V(0,t)] - \frac{1-e^{-a(T-t)}}{a}x_t - \frac{1-e^{-b(T-t)}}{b}y_t.$$

Note.

One might think, at first sight, that in order to implement the G2++ model we need to derive the *whole* ϕ curve, and so the market instantaneous forward curve

$$T \mapsto f^M(0,T).$$

Now, this curve involves *differentiating* the market discount curve

$$T \mapsto P^M(0,T),$$

which is usually obtained from a finite set of maturities via *interpolation*. Now interpolation (obtaining a curve from its values at a finite set of points – there are various ways of doing this; the relevant subject is *Numerical Analysis*) involves a degree of approximation. Again from Numerical Analysis: numerical differentiation is a dangerous process (integration is a smoothing process, which gains accuracy; differentiation makes things rougher, which loses accuracy). Interpolation *followed by* numerical differentiation is asking for trouble.

But, in fact we do not need the whole ϕ curve here. Instead, what we actually need is the *integral* of ϕ between two time-points – and we have calculated this above. From this expression, we see that the only curve needed is the market discount curve, which need not be differentiated, and only at times corresponding to the maturities of the bond prices and rates desired, thus limiting also the need for interpolation. This is a good illustration of how, although the mathematics of interest rates is in principle infinite-dimensional, in practice we can often confine ourselves to finite-dimensional situations, corresponding to the tenor structure – of bonds etc. actually traded in the market. We shall use this systematically in Ch. V below on *market models*.

Short-rate distribution and probability of negative rates

By fitting the currently-observed term structure of discount factors, we find that the expected instantaneous short rate at time t is

$$\begin{aligned} \mu_r(t) &:= E[r_t] \\ &= f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \frac{\eta^2}{2b^2}(1-e^{-bt})^2 + \rho\frac{\sigma\eta}{ab}(1-e^{-at})(1-e^{-bt}), \end{aligned}$$

while the variance $\sigma_r^2(t)$ of the spot rate at t is

$$\begin{aligned} \sigma_r^2(t) &:= var(r_t) \\ &= \frac{\sigma^2}{a^2}(1-e^{-2at}) + \frac{\eta^2}{b^2}(1-e^{-2bt}) + 2\rho\frac{\sigma\eta}{ab}(1-e^{-(a+b)t}). \end{aligned}$$

This implies that the risk-neutral probability of negative rates at time t is

$$\mathbb{Q}(r_t < 0) = \Phi(-\mu_r(t)/\sigma_r(t)),$$

with $\Phi = N(0,1)$ the standard normal distribution function, as usual. This is often negligible – as one would expect, as interest rates are traditionally *positive*!

Note.

If one tries to use the G2++ model, as here, after the beginning of the Crash of 2007, one often finds that the probability of negative interest rates has increased dramatically. This is because the *interest rates* μ observed since then have been low (unprecedentedly low historically – Ch. I), and *volatilities* σ have been high (again, unprecedentedly high). So μ/σ has been low, as it reflects *both* of these. Now $\Phi(0) = \frac{1}{2}$ (by symmetry, and $\Phi(x) \downarrow 0$ as $x \to -\infty$. So $\Phi(-x)$ is small for large x, but large ($\uparrow \frac{1}{2}$) for small x. Thus the formula above reflects the reality of 2007 and after – which has changed everything: world politics, the world economy, the world financial system – and our subject of Interest Rates in particular.

Limit distributions

Like the Ornstein-Uhlenbeck (OU) model to which it is related, the G2++ model has a limit (stationary, equilibrium, ergodic) distribution as time $t \to \infty$. The limit law is Gaussian, with mean and variance given by

$$\mu_r(\infty) := \lim_{t\to\infty} E[r_t] = f^M(0,\infty) + \frac{\sigma^2}{2a^2} + \frac{\eta^2}{2b^2} + \rho\frac{\sigma\eta}{ab},$$

$$\sigma^2_\infty := \lim_{t\to\infty} var(r_t) = \frac{\sigma^2}{2a} + \frac{\eta^2}{2b} + 2\rho\frac{\sigma\eta}{a+b}.$$

Volatility and correlation in 2-factor models

We now derive the dynamics of forward rates under the risk-neutral measure. This gives an equivalent formulation of the two-factor additive Gaussian model in the Heath-Jarrow-Morton (HJM) framework. In particular, we explicitly derive the volatility structure of forward rates. This also shows us which market-volatility structures can be fitted by the model.

Define $A(t,T)$ and $B(z,t,T)$ by

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\{\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]\},$$

$$B(z,t,T) := \frac{1 - e^{-z(T-t)}}{z}.$$

So

$$P(t,T) = A(t,T)\exp\{-B(a,t,T)x_t - B(b,t,T)y_t\}.$$

The forward rates are given by

$$\begin{aligned} f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\ &= -\frac{\partial}{\partial T}\log A(t,T) + \frac{\partial B}{\partial T}(a,t,T)x_t + \frac{\partial B}{\partial T}(b,t,T)y_t, \end{aligned}$$

which in differential form is

$$df(t,T) = [\cdots]dt + \frac{\partial B}{\partial T}(a,t,T)\sigma dW_1(t) + \frac{\partial B}{\partial T}(b,t,T)\eta dW_2(t).$$

So

$$\begin{aligned} \frac{var(df(t,T))}{dt} &= \Big(\frac{\partial B}{\partial T}(a,t,T)\sigma\Big)^2 + \Big(\frac{\partial B}{\partial T}(b,t,T)\eta\Big)^2 + 2\rho\sigma\eta\frac{\partial B}{\partial T}(a,t,T)\frac{\partial B}{\partial T}(b,t,T) \\ &= \sigma^2 e^{-2a(T-t)} + \eta^2 e^{-2b(T-t)} + 2\rho\sigma\eta e^{-(a+b)(T-t)} \end{aligned}$$

(recall that for random variables X, Y,

$$var(X+Y) = var(X) + 2cov(X,Y) + var(Y).)$$

So the volatility (standard deviation (SD) – square root of the variance above) of the instantaneous forward rate is

$$\sigma_f(t,T) = \sqrt{\sigma^2 e^{-2a(T-t)} + \eta^2 e^{-2b(T-t)} + 2\rho\sigma\eta e^{-(a+b)(T-t)}}.$$

Now (as we saw before) a humped-shaped volatility structure is commonly observed in the market for caplets etc. We see immediately from this that desirable feature – a humped-shaped curve – can *only* be produced in this model for *negative* correlation ρ. So when calibrating this model to the market, we need $\rho < 0$. Not all such cases work, but there do exist such choices of parameter values that do.

Options in G2++

Given current time t and future times $T_1 < T_2$, a *caplet* pays off at time T_2

$$[L(T_1,T_2) - X]_+\alpha(T_1,T_2)N,$$

where N is the nominal value ('N for nominal' – *not* for normal, here), $\alpha(T_1, T_2)$ is the year fraction between times T_1 and T_2, X is the strike and $L(T_1, T_2)$ is the LIBOR rate at time T_1 for maturity T_2, i.e.

$$L(T_1, T_2) = \frac{1}{\alpha(T_1, T_2)}[\frac{1}{P(T_1, T_2)} - 1].$$

Writing

$$X^* := \frac{1}{1 + X\alpha(T_1, T_2)}, \qquad N^* = N(1 + X\alpha(T_1, T_2)),$$

we have

$$\begin{aligned} Cpl(t, T_1, T_2, N, X) &= E[D(t, T_2)(L(T_1, T_2) - X)_+ \alpha(T_1, T_2)N] \\ &= -N^* P(t, T_2)\Phi\Big(\frac{\log(NP(t, T_1)/N^* P(t, T_2))}{\Sigma(t, T_1, T_2)} - \frac{1}{2}\Sigma(t, T_1, T_2)\Big) \\ &\quad + NP(t, T_1)\Phi\Big(\frac{\log(NP(t, T_1)/N^* P(t, T_2))}{\Sigma(t, T_1, T_2)} + \frac{1}{2}\Sigma(t, T_1, T_2)\Big), \end{aligned}$$

where $\Sigma(t, T, S)^2$ is a sum of three terms,

$$\Sigma_1^2 := \frac{\sigma^2}{2a^3}[1 - e^{-a(S-T)}]^2[1 - e^{-2a(T-t)}],$$

$$\Sigma_2^2 := \frac{\eta^2}{2b^3}[1 - e^{-b(S-T)}]^2[1 - e^{-2b(T-t)}],$$

$$\Sigma_3^2 := 2\rho\frac{\sigma\eta}{ab(a+b)}[1 - e^{-a(S-T)}][1 - e^{-b(S-T)}][1 - e^{(a+b)(S-T)}].$$

From caplets one gets caps by adding up. Floorlets and floors are completely analogous. For the details, see Brigo & Mercurio [BM, §4.2.4].

4. What do we measure? What should we measure? What is random?

The situation here is (though more complicated) rather like that with stock prices in MATL480, but there are differences. Prices evolve randomly; at time t – 'now' – we know the current stock price S_t. The risk-free interest rate r of MATL480 (constant, non-random and known there) corresponds to the spot rate, the *stochastic process* $r = (r_t)$ here in MATL481. We can't measure this directly; what we can measure is *bond prices* $P(t,T)$. Recall:

$$P(t,T) = E_t[\exp\{-\int_t^T r_s ds\}]. \qquad (P,r)$$

So r is inaccessible here: it is linked with $P(t,T)$, which we can see, but hidden behind both an expectation (E_t is conditional expectation, given what we know now at time t, over the uncertainty over the relevant time-interval $[t,T]$ extending into the future), and an integration.

Note that the bond price $P(t,T)$ above is non-random, in one sense: we know it at time t; it's the price that ZCBs with maturity T are selling at. But it is random, in another sense (see below): it is a *conditional expectation*, and (MATL480) a conditional expectation is random, as it is a function of what we are conditioning on, and that is itself random.

There is much to be said here, but for simplicity: we will generally use 'random' to mean things depending on the still-uncertain future, and 'known' to mean things we know now.

Randomness

Regarding randomness, recall again the simpler case of stock prices $S = (S_t)$. Future stock prices are uncertain, because we live in an uncertain world. Past stock prices are known – we can look them up; but they are still random, in the sense that they once were (as they were still in the future once). Of course, all this goes back to the basics of Probability and Statistics. In Probability, we need to know the mechanism generating the randomness (the relevant distributions, or models); we can then carry out the necessary calculations (e.g., of the probability that this or that will happen). In Statistics, the basic raw material is *data*; we seek to use the information in the data to infer what the underlying mechanism generating it is. Think of tossing a coin, ten times, say. While the coin is still in the air spinning, the value it comes down as (1 for head, 0 for tail) is still unknown; the outcome is a *random variable.* When the coins have fallen, the outcomes are data (ten-tuples of 0s and 1s, in this case). They are known (in that we have them written down as numbers); they are *realised values* of random variables. We can think of this (the essence of the interplay and contrast between the twin subjects of Probability and Statistics, by the way) as 'fossilised randomness': they are known (above); they *could* have been different; they *would* be different if we did it again. We see this all the time – e.g., in ourselves. We inherit our genes – from our parents/God/Mother Nature; we could have been different (indeed, our siblings are – even with identical twins!); we're not – we're stuck with what we've got.

By contrast, the forward rate $f(t,T)$ is non-random (in the sense above: it's a function of the bond prices $P(t,T)$, known at time t – "now"). It is given by

$$P(t,T) = \exp\{-\int_t^T f(t,u)du\}, \quad f(t,T) = -\frac{\partial}{\partial T}\log P(t,T). \qquad (P,f)$$

There is no expectation here, but we still can't get at f directly: it involves a differentiation wrt time T in the future. We do have bond prices $P(t,T)$ involving T, but we only have them for a discrete set of Ts, the T_i (given by the tenor structure), and so differentiation (which would have to be done numerically) can be done only in theory (as in (P,f) above) and not in practice.

What is needed is an approach in which we focus on *things we can see and measure* – the $P(t,T)$ – and not on the spote-rates r_t and forward-rates $f(t,T)$ above. For this we need the *market models*, which are the subject of Ch. V (W4ab,5ab,6a) in Part II of this course, below.

Rates, and measuring them.

It is possible to measure rates, as is done on vehicle speedometers (via a cable from the front wheel), speed cameras (for monitoring vehicle speeds in restricted areas), etc. But this needs special equipment, and is subject to measurement error (as are all measurements – except *counts*). This is one reason why the spot rates r_t of Ch. III and the forward rates $f(t,T)$ of Ch. IV are unsatisfactory: it is difficult to measure them. Indeed, we cannot measure them directly; what we can measure is some *proxy* for them. As we saw in I.1 Note 2 (W1a) and II.1 (W2a), LIBOR is the one most used now, but as it has proved problematic (because of market manipulation), it is planned to be phased out and replaced by Sonia.

Market models (preview).

The problems concerning *rates* (above), which are *instantaneous*, can be avoided by dealing with things that relate to *time-intervals*. For example, LIBOR relates to $[t,T]$, with t 'time now', and T the end of the time-interval for which the rate is being quoted. Similarly, forward LIBOR deals with time t now *and* a time-interval $[T,S]$ in the future. These determine the prices of interest-rate derivatives, such as the forward-rate agreements (FRAs) in Ch. II. These are highly liquid (heavily traded): there are many such (hundreds), but the amounts of money involved (trillions) are so huge that we know their prices. We don't have to model them: we can see them. As we shall see (Ch. V), calibration of market models to such market data is largely a matter of *correlations* (between rates for different time-intervals), and *volatilities* (reflecting market uncertainty, as with stock prices).