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Part 2

V. MARKET MODELS

1. Introduction

Market models in a nutshell: *Don't try to model infinite-dimensional things you can't see. Model instead finite-dimensional things you can see.*

Before market models were introduced (in 1997), short-rate models (III) were the main choice for pricing and hedging of interest-rate derivatives. They are still used for many applications, and are based on modelling the instantaneous short (or spot) rate r_t via a (perhaps multidimensional) diffusion process. This diffusion characterises the time-evolution of the complete yield curve. Short-rate models were followed by forward-rate models (IV).

It is better to model what one can actually *see*. This is the prices at which liquid products are traded, in the *market. This is what market models do.* One cannot actually *see* forward rates and short rates.

What makes all this work is that, although interest rates are in principle *infinite-dimensional* – the yield curve, or term-structure of interest rates, is an infinite-dimensional object – because only finitely many products are traded in the market (which ones are determined by the tenor structure), and these are highly liquid, all we really need is to model these. In practice, this largely reduces to modelling two *finite-dimensional* things: the *correlations* (V.9), and the *volatilities* (V.13).

The introduction of market models in 1997 was one of two things that changed the whole nature of interest rates, in both theory and practice. The other was the Crash of 2007(-08). So it is worth checking the *date* of any source you consult on interest rates. If it's more than two decades old: use for background and interest, rather than as a primary source of information.

Market models were introduced by Brace, Gatarek and Musiela (1997)[1], Miltersen, Sandmann and Sondermann (1997) and Jamshidian (1997) – independently, and in the same year. Rather than use all seven names, the term *market model* is used, as this captures the essence: Model what we can *see* in the *markets*.

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[1]Musiela, a Polish name: 'Mu*shell*a' – 'si' in Polish is pronounced 'sh', as in Welsh – Siân, Moel Siabod etc. Again as in Welsh: stress on the *penultimate* syllable.

Humped volatility

Furthermore, there are some aspects of visible market data that can be modelled successfully by market models (below), but not always, or not so easily, by the models of III, IV. A good example is the *term structure of volatility* – the way volatility varies with time (we return to volatility modelling in V.13 below). This is typically observed in the market to have a *humped* shape – the curve increases at first (reflecting that the near future is more predictable than the intermediate future), but then decreases (reflecting a 'discounting' of the far future). Often the hump is observed about 5-6 years into the future. The ability to model this characteristic humped shape is a good measure of the suitability of the model. For background here, see [BM] (index, Humped volatility, ten references), and the papers

[MM1] F. MERCURIO and J. M. MORALEDA, An analytically tractable interest rate model with humped volatility. *European J. Operational Research* **120** (2000), 205-214;

[MM2] F. MERCURIO and J. M. MORALEDA, A family of humped volatility models. *European J. Finance* **7** (2001), 93-116.

To introduce market models, recall the forward LIBOR rate at time t between T and S (II.2 p7, W2a),

$$F(t; T, S) = \frac{1}{(S-T)}(P(t,T)/P(t,S) - 1),$$

which makes the FRA contract to lock in at time t the interest rates between T and S fair (have cost/value zero). **A family of such rates for** $(T, S) = (T_{j-1}, T_j)$ **spanning** $T_0, \cdots, T_M$ **is modelled in the LIBOR market model (LMM)**. We set

$$F_j(t) := F(t; T_{j-1}, T_j); \qquad \tau_j := T_j - T_{j-1} :$$

$$F_j(t) = \frac{1}{\tau_j}\Big(\frac{P(t, T_{j-1})}{P(t, T_j)} - 1\Big); \qquad \frac{P(t, T_j)}{P(t, T_{j-1})} = \frac{1}{1 + \tau_j F_j(t)}. \qquad (F_j)$$

With the LMM, we may specify precise *volatilities* and *correlations*:

$$\sigma_j \leftrightarrow F_j; \qquad \rho_{j-1,j} \leftrightarrow F_j \leftrightarrow (T_{j-1}, T_j), \qquad \rho_{j,k} \leftrightarrow (T_{j-1}, T_k).$$

These are rates associated to market payoffs – FRAs – and not abstract rates such as r_t or $f(t, T)$ (rates on infinitesimal maturities/tenors.

Taking $t = T_1$ and using $P(T_1, T_1) = 1$ gives

$$F_2(T_1) = \frac{1}{T_2 - T_1}\Big(\frac{P(T_1, T_1)}{P(T_2, T_1)} - 1\Big) = \frac{1}{T_2 - T_1}\Big(\frac{1}{P(T_2, T_1)} - 1\Big).$$

But from II.1, $(P - L)$ (W2a),

$$P(T_1, T_2)(1 + (T_2 - T_1)L(T_1, T_2)) = 1 : L(T_1, T_2) = \frac{1}{T_2 - T_1}\Big(\frac{1}{P(T_2, T_1)} - 1\Big).$$

Comparing,

$$L(T_1, T_2) = F_2(T_1).$$

To further motivate market models, let us consider the time-0 price of a T_2-maturity caplet resetting at time T_1 $(0 < T_1 < T_2)$ with *strike* (rate here, not price as in MATL480) K and a notional amount of 1. Write τ for the *year-fraction* between T_1 and T_2. The contract pays out at time T_2 the amount

$$\tau(L(T_1, T_2) - K)_+ = \tau(F_2(T_1) - K)_+.$$

On the other hand, the market has been pricing caplets (actually caps) with *Black's caplet formula* (V.2 below) for years. We now derive this formula rigorously under the LIBOR model dynamics, the only dynamical model that is consistent with it.

FACT 1.

The price of any asset divided by a reference asset (numeraire) is a martingale (no drift) under the measure associated with ('for') that numeraire.

In particular,

$$F_2(t) = \frac{(P(t, T_1) - P(t, T_2))/(T_2 - T_1)}{P(t, T_2)}$$

is a portfolio of two ZCBs divided by the ZCB $P(., T_2)$. If we take the measure $\mathbb{Q}_2$ associated with this numeraire, $P(., T_2)$, then by FACT 1 F_2 will be a $\mathbb{Q}_2$-martingale (mg) – no drift – under that measure:

F_2 is a mg (no drift) under the measure $\mathbb{Q}_2$ for the numeraire $P(., T_2)$.

FACT 2.

The time-t risk-neutral price

$$\text{Price}_t = E_t^B\Big[B_t.\frac{\text{Payoff}(T)}{B(T)}\Big]$$

3

(under the numeraire B – 'B for bank account') is invariant under change of numeraire $B \mapsto S$: for any other numeraire S, we have

$$\text{Price}_t = E_t^S[S_t . \frac{\text{Payoff}(T)}{S(T)}].$$

That is, if we substitute S for B in all three places above, the price does not change. This is the Numeraire Invariance Theorem; see e.g. [BK, Prop. 6.1.1] for a formal proof. We omit the proof here, as the result is so intuitive: it does not matter whether we reckon in pounds, dollars, euros etc. – nothing important changes.

2. Black's caplet formula

This classic result – due to Black (1976) – is a variant on the Black-Scholes formula of 1973; see e.g. [BM, 6.2, p.200-202, 6.4], [BK, 8.5.4].
Vega.
Recall (MATL480) that vega (the partial derivative of the price wrt the volatility) is *positive*. So price and volatility are continuous strictly increasing functions of each other ('options like volatility'). So one price corresponds to one volatility ('implied volatility': 'Black-Scholes vol'). Similarly here, giving the 'Black volatility' – *Black vol* – for caplets ($v_1(T_1)$ in Black's caplet formula below). As with the Black-Scholes formula, the main use of Black's caplet formula is to allow traders to find the *implied volatility*.
Consider now the caplet price, and apply FACT 2: replacing B by $P(., T_2) =: P(., 2)$ as numeraire, and writing

$$F_2(t) := F(t; T_1, T_2), \qquad \tau := T_2 - T_1$$

as usual, the caplet price is (writing $E_{\mathbb{Q}_2}$ as E_2)

$$E^B[\frac{B(0)}{B(T_2)}\tau(F_2(T_1)-K)_+] = E_{\mathbb{Q}_2}[\frac{P(0,T_2)}{P(T_2,T_2)}\tau(F_2(T_1)-K)_+] = P(0,T_2)\tau E_2[(F_2(T_1)-K)_+]$$

(as $P(0, T_2)$ is known at time 0 and $P(T_2, T_2) = 1$). By FACT 1, F_2 is a $\mathbb{Q}_2$-mg (no drift). Take a geometric Brownian motion (GBM), with initial condition (IC) the observed market rate at time 0,

$$dF(t; T_1, T_2) = \sigma_2(t)F(t; T_1, T_2)dW_2(t) : \ dF_2(t) = \sigma_2(t)F_2(t)dW_2(t),$$
$$(LMM)$$

4

$$\text{IC } mkt \ F(0; T_1, T_2),$$

(no drift! – by above), where σ_2 is the instantaneous volatility, and W_2 is BM under $\mathbb{Q}_2$. **The forward LIBOR rates, the Fs, are the quantities that are modelled in the LIBOR market model (LMM), instead of r and f**.

We now prove the classic *Black's caplet formula*:
Fischer BLACK, The pricing of commodity contracts. *J. Financial Economics* **3** (1976), 167-179.
Write

$$v_1(T_1)^2 := \frac{1}{T_1} \int_0^{T_1} \sigma_2(t)^2 dt$$

– so $v_1(T_1)$ is the time-averaged quadratic volatility.

Theorem (Black's caplet formula). The price of the caplet is

$$
\begin{aligned}
Cpl(0, T_1, T_2, K) \ &:= \ P(0, T_2)\tau E[(F(0; T_1, T_2) - K)_+] \\
&= \ P(0, T_2)[F_2(0)\Phi(d_+) - K\Phi(d_-)],
\end{aligned}
$$

where

$$d_\pm = \frac{\log(F_2(0)/K) \pm \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1} v_1(T_1)}.$$

Proof. We solve the SDE (LMM) above, and compute $E_2[\tau(F_2(T_1) - K)_+]$. By Itô's formula, as $\log' x = 1/x$, $\log'' x = -1/x^2$, $(dW_2(t))^2 = dt$, the SDE (LMM) gives

$$
\begin{aligned}
d\log F_2(t) \ &= \ \frac{1}{F_2}dF_2 + \frac{1}{2}\left(-\frac{1}{F_2^2}\right)dF_2 dF_2 \\
&= \ \frac{1}{F_2}\sigma_2 F_2 dW_2 + \frac{1}{2}\left(-\frac{1}{F_2^2}(\sigma_2 F_2 dW_2)^2\right) = \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2 dt :
\end{aligned}
$$

$$d\log F_2(t) = \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2 dt.$$

Integrate both sides:

$$\log F_2(T) - \log F_2(0) = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt :$$

$$F_2(T) = F_2(0)\exp\{\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt\}.$$

The distribution of the random variable in the exponent is Gaussian, since it is a stochastic integral of a deterministic function by a Brownian motion (MATL480 Problems 5b Q1 – sums of independent Gaussians are Gaussian). Compute its expectation: as the Itô integral has mean 0,

$$E[\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt] = -\frac{1}{2}\int_0^T \sigma_2(t)^2 dt.$$

The variance is

$$
\begin{aligned}
var(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt) &= var(\int_0^T \sigma_2(t)dW_2(t)) \\
&= E[(\int_0^T \sigma_2(t)dW_2(t))^2] \quad \text{(as the mean is 0)} \\
&= \int_0^T \sigma_2(t)^2 dt,
\end{aligned}
$$

(by Itô's isometry: MATL480, V.5). Summarising,

$$I(T) := \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt \sim m + VN(0,1)$$

(here '$\sim m + VN(0,1)$' is shorthand for 'is distributed as $m + V$ times a $N(0,1)$ – a standard normal random variable'), where

$$m = -\frac{1}{2}\int_0^T \sigma_2(t)^2 dt, \qquad V^2 = \int_0^T \sigma_2(t)^2 dt \qquad (\text{so } m + \frac{1}{2}v^2 = 1).$$

That is,

$$F_2(T) = F_2(0)\exp\{I(T)\} = F_2(0)e^{m+VN(0,1)},$$

in an obvious extension of the above shorthand notation. Now compute the option price: with $T = T_1$ in the above,

$$
\begin{aligned}
E_2[(F_2(T_1) - K)_+] &= E_2[(F_2(0)e^{m+VN(0,1)} - K)_+] \\
&= \int_{-\infty}^{\infty} (F_2(0)e^{m+Vy} - K)_+ . \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} dy.
\end{aligned}
$$

The rest of the calculation resembles that of the proof of the Black-Scholes formula! (predictably, as Black's caplet formula is obviously an extension of the Black-Scholes formula). In the integrand, $[...] > 0$ iff

$$y > \bar{y} := \frac{-\log(F_2(0)/K) - m}{V}.$$

6

Here

$$-\overline{y} \;=\; \frac{\log(F_2(0)/K) + m}{V} = \frac{\log(F_2(0)/K) - \frac{1}{2}\int_0^{T_1}\sigma_2^2}{\sqrt{\int_0^{T_1}\sigma_2^2}}$$

$$=\; \frac{\log(F_2(0)/K) - \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1}v_1(T_1)} = d_2,$$

$$V - \overline{y} \;=\; \sqrt{T_1}v_1(T_1) + \frac{\log(F_2(0)/K) - \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1}v_1(T_1)}$$

$$=\; \frac{\log(F_2(0)/K) + \frac{1}{2}T_1 v_1(T_1)^2}{\sqrt{T_1}v_1(T_1)} = d_1.$$

So the RHS above is

$$\int_{\overline{y}}^{\infty}(F_2(0)e^{m+Vy} - K).\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy \;=\; F_2(0)\int_{\overline{y}}^{\infty}e^{m+Vy}.\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy - K\int_{\overline{y}}^{\infty}\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy$$

$$=\; F_2(0)I_1 - KI_2,$$

say. Completing the square (cf. the proof of Black-Scholes!), as $m+\frac{1}{2}V^2 = 1$,

$$I_1 \;=\; \int_{\overline{y}}^{\infty}e^{m+Vy}.\frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}dy = \int_{\overline{y}}^{\infty}\exp\{-\frac{1}{2}(y-V)^2 + m + \frac{1}{2}V^2\}dy/\sqrt{2\pi}$$

$$=\; \int_{\overline{y}-V}^{\infty}\frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}dz \qquad (z := y - V)$$

$$=\; (1 - \Phi(\overline{y} - V)) = \Phi(V - \overline{y}) = \Phi(d_+),$$

$$I_2 = \Phi(-\overline{y}) = \Phi(d_-).$$

Combining,

$$E_2[(F_2(T_1) - K)_+] = F_2(0)\Phi(d_+) - K\Phi(d_-). \qquad //$$

Corollary. Vega, defined w.r.t. the *Black vol* $v_1(T_1)$, is positive.

Proof. The same proof as for the Black-Scholes vega works. //

The market converts caplet prices Cpl to volatilities and vice-versa. The v_1 here is the *Black volatility* (*Black vol* for short); see V.6 (W4b, p.7) below. Here as with BS, we used *implied vol*: one can see market prices; one can't see vol directly. Here, as in BS (MATL480, V), the key is the SDE for GBM (in (LMM) above), and *log-normality*.

Black's caplet formula (and *Black's swaption formula* – V.4 below) were used by traders in the market long before rigorous proofs for them were known. The models used to derive the two are in fact incompatible – see e.g. [BM, 6.8] for details. Nevertheless, each is used, and traders are happy with each in its proper context. See e.g. [BM, 6.17, p287-288].

3. LIBOR Market Models (LMM)
Model calibration

Our raw material is the prices of standard products that are *liquid* – freely traded in the market, so that we know (fairly closely) what they are worth (basically, what the market thinks they are worth – what a seller can get for them and a buyer can get them for) – for example, FRAs, swaps, caps, swaptions etc. We use the above model to find, for each caplet maturity T_2 and strike K that interests us, the volatility σ_2 that matches the market price of the caplet when plugged into the above Black formula.

Note that Black found his formula in 1976, 21 years before it was proved rigorously. Of course, he was helped and guided by the analogy with the Black-Scholes formula of 1973. But this remains a remarkable feat. Fischer Black was the genius of Black-Scholes(-Merton) (and the only one who did not get a Nobel Prize, having died too early).

Recall (MATL480) that the Black-Scholes model itself is not exact (indeed, Black himself famously wrote a paper called *The holes in Black-Scholes*), but nevertheless the Black-Scholes formula is widely used in practice. It is close enough to reality to be useful as a benchmark, enabling traders to *calibrate* their models to data. Similarly for Black's caplet formula.

Black's formula and spot-rate models

Can the model above leading to Black's caplet formula be obtained as a special *spot-rate* (short-rate) model? That is, is there an SDE for r that is consistent with the Black caplet formula, i.e. with the lognormal distribution of the Fs? To fix ideas, can we use a Vasicek model?

$$r_t = x_t, \qquad dx_t = \kappa(\theta - x_t)dt + \sigma dW_t.$$

Such a model allows for an analytical formula for forward LIBOR rates F,

$$F(t; T_1, T_2) = F_{VAS}(t; T_1, T_2; x_t, \alpha), \qquad \alpha = (\kappa, \theta, \sigma).$$

To price a caplet under this model is to compute the risk-neutral expectation

$$E[\frac{B(0)}{B(T_2)} \tau (F(0; T_1, T_2, X_{T_1}, \alpha) - X)_+].$$

This can be done, and leads to a function,

$$U_{VAS}^C(0; T_1, T_2, X, \alpha),$$

say. But this does not lead to Black's formula; F_{VAS} is not lognormal; nor are Fs associated with other known short-rate models. So: *no known short-rate model is consistent with the market formula.* Short-rate models are calibrated through their particular formulas for caplets, but these are not Black's market formula (though some are close).

When the Hull-White (extended Vasicek) model is calibrated to caplets, there are values of $\kappa, \theta, \sigma, x_0$ consistent with caplet prices – but these parameters do not have an *immediate intuitive meaning* for traders, who thus do not know *how to relate them to Black's market formula*. By contrast, the parameter σ_2 in the market model *has an immediate meaning as the Black caplet volatility (Black vol) of the market. There is an immediate link between model parameters and market quotes. Language is important.*
Several caplets

With several caplets involving different forward rates,

$$F_2(t) = F(t; T_1, T_2), \quad F_3(t) = F(t; T_2, T_3), \ \cdots \ F_k(t) = F(t; T_{k-1}, T_k),$$

or with swaptions, one can use different structures of instantaneous volatilities. One can select a different σ for each forward rate, by assuming each forward rate to have a constant instantaneous volatility. Moreover, different forward rates can be modelled as having different sources of randomness that are *instantaneously correlated*. So we have great freedom, and can model

$$corr(dF_i(t), dF_j(t)) = \rho_{ij} dt,$$

whereas in one-factor short-rate models dr these correlations were fixed to be 1. Modelling correlation is necessary for pricing payoffs depending on more

than a single rate at a given time, such as swaptions.

The dynamics of
$$F_k(t) = F(t; T_{k-1}, T_k)$$
under $\mathbb{Q}_k$ (numeraire $P(., T_k)$) is

$$dF_k(t) = \sigma_k(t)F_k dZ_k(t),$$

leading to the lognormal distribution – all as in the example above with $k = 2$. *However*, the dynamics of F_k under $\mathbb{Q}_i \neq \mathbb{Q}_k$ for i, k not adjacent is more involved. The drift (local mean) is complicated, and does not lead to a known distribution of F_k. So, the model needs to be used with simulations (MATL484) – no PDEs, or approximations (drift freezing).

Precisely because the $\mathbb{Q}_k$-dynamics of $F_k(t) = F(t; T_{k-1}, T_k)$ is $dF_k(t) = \sigma_k(t)F_k dZ_k(t)$, lognormally distributed, the LIBOR market model is calibrated to caplets *automatically* through integrals of the squared deterministic functions $\sigma_k(t)$. For example, if one takes constant $\sigma_k(t) \equiv \sigma_k$, then σ_k is the market caplet volatility for the caplet resetting at T_{k-1} and paying at T_k.

As with Black-Scholes (vega, above), there is no problem in inverting the Black formula – 'reverse-engineering' – to find the σ_2 matching the LIBOR-model caplet price to the caplet price observed in the market. By contrast, it is complicated to do this for Vasicek-model caplet prices.

Swaptions can be calibrated under good approximations, and the swaptions market formula is almost compatible with the model.

Pros and cons of the LIBOR market model for Fs

Pro. The advantages of the LMM for Fs include:

(a) immediate and intuitive (for traders) calibration of caplets (better than any short-rate models);

(b) easy calibration to swaptions through algebraic approximation (again, better than most short-rate models);

(c) can calibrate a large number of market products, exactly or with a precision impossible with short-rate models;

(d) clear correlation parameters, as these are instantaneous correlations of market forward rates;

(e) powerful disgnostics: can check *future* volatility and terminal correlation structures (diagnostics are impossible with most short-rate models);

(f) can be used for Monte Carlo simulation (MATL484).

Con. Limitations include:

(a) high dimensionality – many F evolving jointly (awkward: see (b) below);

(b) unknown joint distribution of the F (although each is lognormal under its canonical measure); this is why we rely so much on *correlations*;
(c) difficult to use with PDEs or lattices/trees (but recent Monte-Carlo approaches such as Least-Squares MC makes trees and PDEs less necessary).

4. Swap Market Models (SMM). Black's swaption formula

The LIBOR market model is not the only market model. The simple market options on interest rates split into two markets, *caps/floors* and *swaptions*; each is traded in *enormousl volumes*. The LMM is the model of choice for caplets, as we have seen, since it produces the Black-Scholes type (Black's) caplet formula the market uses to quote implied volatilities.

But what about swaptions? These can be managed well using LMM only through approximations such as drift freezing. To deal with swaptions properly, we have to use a different market model, the swap market model (SMM). We now present this briefly.

Consider the payer swaption giving the right (not obligation) to enter into a swap first resetting at T_i and paying at $T_{i+1}, \cdots T_k$, for a fixed rate K.

Recall that we can write the payoff of such an option at maturity T_i as

$$(S_{ik}(T_i) - K)_+ \sum_{j=i+1}^{k} \tau_j P(T_i, T_j).$$

Define the annuity numeraire, also called the Present Value per Basis Point (PVPBP), PV01, and the related measure:

$$U = C_{i,k}(t) := \sum_{j=i+1}^{k} \tau_j P(T_i, T_j), \qquad \mathbb{Q}_U = \mathbb{Q}_{ik}.$$

By FACT 1 the forward swap rate S_{ik} is then a $\mathbb{Q}_{ik}$-mg:

$$S_{ik}(y) = \frac{P(t, T_i) - P(t, T_k)}{\sum_{j=i+1}^{k} \tau_j P(t, T_j)} = \frac{P(t, T_i) - P(t, T_k)}{C_{ik}(t)}.$$

Take the usual mg (zero-drift) lognormal GBM,

$$dS_{ik}(t) = \sigma_{ik}(t) S_{ik}(t) dW_{ik}(t), \qquad \text{(under } \mathbb{Q}_{ik}). \qquad (SMM)$$

By FACT 2 on change of numeraire, we obtain the following well-known Black's formula for swaptions (see e.g. [BM, 6.7], [BK, 8.5.5]):

Theorem (Black's formula for swaptions).

$$E_B[(S_{ik}(t) - K)_+ C_{ik}(T_i).\frac{B(0)}{B(T_i)}] = E_{ik}[(S_{ik}(t) - K)_+ C_{ik}(T_i).\frac{C_{ik}(0)}{C_{ik}(T_i)}]$$
$$= C_{ik}(0)E_{ik}[(S_{ik}(t) - K)_+]$$
$$= C_{ik}(0)[S_{ik}(0)\Phi(d_1) - K\Phi(d_2)],$$

$$d_{1,2} = \frac{\log(S_{ik}(0)/K) \pm T_i v_{ik}(T_i)^2}{\sqrt{T_i} v_{ik}(T_i)}, \qquad v_{ik}(T)^2 = \frac{1}{T}\int_0^T \sigma_{ik}(t)^2 dt.$$

Like Black's formula for caplets above, it is a formula of Black-Scholes type. It is how the market converts swaption prices into swaption-implied volatilities. SMM is the only model that is consistent with this market formula. But, LMM is not compatible with the Black formula for swaptions.

The SMM is not used as much as the LMM. The reason is that swap rates do not recombine as well as forward rates in describing other rates. Also, swaptions can be priced easily in LMM through drift-freezing with formulas that are very similar to the market swaptions formula. So, even if in principle the two models are not compatible or consistent, in practice the LMM is quite close to the SMM even in terms of swap-rate dynamics.

So we will focus on the LMM below.

Swap curves and Principal Components Analysis (PCA)

We give an example [LX, 2.2.3]. Lai and Xing there consider the swap rates between US Treasury and LIBOR swap rates, over a 5-year period. They use *principal components analysis (PCA)* from Statistics (see e.g. SMF, III.5), used to break down variability into components. They conclude that:
(i) The first principal component (PC) accounts for 90.8 % of the variability. It is a *shift*: a change in the swap rate for one maturity is accompanied by the same change for other maturities.
(ii) The second PC accounts for 6.7 %. It is a *tilt*: changes in short-maturity and long-maturity swap rates have opposite effects.
(iii) The third PC represents 1.3 % of the variability. It is a *curvature* effect, reflecting the typical humped-shape.

This ends our guided tour of the LIBOR model. We now begin the detailed presentation.