PAPER CODE NO.
MATL 481

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DEPARTMENT: Mathematical Sciences

UNIVERSITY OF LIVERPOOL

JUNE 2017 EXAMINATIONS

Interest Rate Theory

TIME ALLOWED: 150 Minutes

INSTRUCTIONS TO CANDIDATES: Full marks will be awarded for correct answers to FIVE questions. Each question carries equal weight.

1. With $\rho := \partial C/\partial r$ the partial derivative of an option price with respect to the riskless interest rate r, $\tau := T - t$,

$$d_1, d_2 := \frac{\log(S/K) + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

as in the Black-Scholes formula for European calls and puts:
(i) Show that

$$Ke^{-r\tau}\phi(d_2) = S\phi(d_1),$$

with ϕ the standard normal density function.
(ii) Show that for calls C, $\rho := \partial C/\partial r > 0$.
(iii) Give the financial interpretation of this.
(iv) Show that for puts P, $\rho := \partial P/\partial r < 0$.
(v) Again, give the financial interpretation of this.
(vi) Do these results extend to American options? If so, why? [20 marks]

2. Define a *forward rate agreement* (FRA) with current time t, expiry time $T > t$, maturity time $S > T$ and strike rate K for *receiver* FRAs and *paper* FRAs.

With $P(T, t)$ the price of a zero-coupon bond at time t with maturity T, show that the price of a receiver FRA is

$$\mathrm{FRA}(t, T, S, K) = P(t, S)(S - T)K - P(t, T) + P(t, S).$$

What is the price of the corresponding payer FRA? [20 marks]

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3. The Ornstein-Uhlenbeck SDE is

$$dV_t = -\kappa V_t dt + \sigma dW_t, \quad V_0 = v_0, \qquad \text{(OU)}$$

with W Brownian motion.
(i) Give the interpretation of the parameters κ, σ.
(ii) Solve (OU) to obtain

$$V_t = v_0 e^{-\kappa t} + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dW_s.$$

(iii) Show that V_t is Gaussian, with mean 0 and variance

$$\text{Var } (V_t) = \frac{\sigma^2(1 - e^{-2\kappa t})}{2\kappa}.$$

(iv) Show that the covariance is

$$\text{cov } (V_t, V_{t+k}) = \frac{\sigma^2 e^{-\kappa t}(1 - e^{-2\kappa t})}{2\kappa}.$$

(v) Show that the process V is Markov, and has a limit as $t \to \infty$ which is stationary Gaussian Markov.
(vi) What is meant by saying V is *mean-reverting*? What is the link with the Vasicek model for the short rate of interest?
(vii) Comment briefly on the limitations of the model.

[20 marks]

4. The Black caplet formula is

$$\text{Cpl}(0, T_1, T_2, K) = P(0, T_2)\tau[F_2(0)\Phi(d_1) - K\Phi(d_2)],$$
$$d_1, d_2 := \frac{\log(F_2(0)/\kappa) \pm T_1 v_1(T_1)^2}{\sqrt{T_1} v(T_1)}, \quad \tau := T_2 - T_1.$$

Using any assumptions you need (which should be clearly stated), show that the density p_2 of $F_2(T_1)$ under the T_2-forward measure is given by

$$p_2(K)\tau\rho(0, T_2) = \frac{\partial^2}{\partial K^2}\text{Cpl}(0, T_1, T_2, K).$$

How would one use this in practice? [20 marks]

5. Comment briefly on:
(i) Corporate bonds;
(ii) Junk bonds;
(iii) Credit rating.
(iv) Credit scoring. [20 marks]