MATL481 EXAMINATION SOLUTIONS 2017-18

Q1 (*Overnight indexed swaps; inflation; deflation; monetary policy*).

Overnight Indexed Swaps (OIS)

These were introduced in the mid-90s. Maturities range from 1 week to 2 years or longer. They are based on the *overnight rates*, used by banks to lend to each other for a day or two. These are harder to manipulate than LIBOR (some are quoted by central banks), and as the loan period is short there is little credit risk.

Nowadays – still in the aftermath of the Crisis of 2007/8 – it is no longer realistic to ignore credit risk and liquidity effects in interest-rate modelling – in effect, pretending that there *is* a risk-free rate governing the LIBOR and inter-bank markets. OIS is a partial solution, as it is the best proxy for a (non-existent) default- and liquidity-free interest rate. But some credit-risk and liquidity effects remain, and show up especially in stress-testing under strong stress scenarios. [**6**]

Inflation

The damaging effects of *inflation* are well-known (e.g., the hyper-inflation in Germany and Austria post-WWI devastated their economies, and so their societies and political systems, and paved the way to the rise of Nazism, so to WWII). Inflation eats up the purchasing power of money one already has – so there is no incentive to save, and those dependent on savings suffer. Because it decreases the purchasing power of wages and salaries, it increases pressure for wage/salary increases (e.g. from trades unions). Such wage increases push up the costs of producing goods, which pushes up prices, so generating more inflation ... The danger is runaway inflation (the UK in the 1970s had annual inflation rates of over 20 %). But inflation benefits mortgage-holders, as it decreases the real value of their debt. [**6**]

Deflation

With Bank Rate so low – down to 0.25 %, ridiculously low by historical standards – and in view of frictional costs, real interest rates are effectively negative. This is potentially very dangerous, because of the risk of *deflation*: if prices are falling, people may defer buying till later, to get things cheaper; the economy will then freeze up even worse, exacerbating the whole problem ... . Negative interest rates have actually happened ... Deflation hits house-owners, as the value of their house may fall, even below their mortgage debt, so leaving them in *negative equity.* [**6**]

Monetary policy

Governments and central banks need to steer a middle course between these two! This is done by *monetary policy* (leaving to one side *fiscal policy* – taxation and government spending), now administered by the Monetary Policy Committee (MPC) of the Bank of England (BoE). Failure to get this balance right may have devastating consequences for the economy, which can have long-lasting effects – witness e.g. Japan's 'lost decade' of the 1990s (now 'lost decades'), and the difficulty of the UK in recovering from the Crash of a decade ago. **[7]**

[Largely seen – lectures]

Q2 (*LIBOR; QE*).

LIBOR. The *spot-LIBOR rate* $L(t,T)$ at time t for maturity T is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t,T)$ units of currency at time t, when accruing occurs *proportionally* to the investment time, with $P(t,T)$ the bond price (price at t of a zero-coupon bond giving 1 at T):

$$P(t,T)(1+(T-t)L(t,T))=1, \qquad L(t,T)=\frac{1-P(t,T)}{(T-t)P(t,T)}. \qquad (P-L)$$

This (the definition of LIBOR!) is an *extremely important relationship!*. For, when one of P and L goes *up*, the other goes *down* – the two are *inversely related*. This is the key to *quantitative easing (QE)* (below). **[12]**

Quantitative easing (QE). From the Bank of England's website:
"What is quantitative easing?

Quantitative easing is when a central bank like the Bank of England creates new money electronically to make large purchases of assets. We make these purchases from the private sector, for example from pension funds, high-street banks and non-financial firms. Most of these assets are government bonds (also known as gilts). The market for government bonds is large, so we can buy large quantities of them fairly quickly.

The purchases are of such a scale that they push up the price of assets, lowering the yields (the return) on them [by $(P-L)$]. This encourages those selling these assets to us to use the money they received from the sale to buy assets with a higher yield instead, like company shares and bonds.

As more of these other assets are bought, their prices rise because of the increased demand. This pushes down on yields in general. The companies that have issued these bonds or shares benefit from cheaper borrowing because of these lower yields, encouraging them to spend and invest more."

QE: Effects. The QE programmes, applied extensively after the Crash of 2007/8, have been broadly successful in keeping interest rates [e.g. LIBOR] low. But QE has had extensive, unanticipated and undesirable side-effects. It increases stock prices; stock is bought by the affluent. So those with money gain; those without 'lose out'. This decreases social mobility, and increases resentment and political tension – with effects such as Brexit. **[13]**

[Seen – lectures]

Q3 (*Theta*). Given

$$Ke^{-r(T-t)}\phi(d_2) = S\phi(d_1): \qquad (*)$$

(i) *Calls*. Given the Black-Scholes formula for the price c_t of European calls,

$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

$$d_{1,2} := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}: \qquad d_2 = d_1 - \sigma\sqrt{T-t}:$$

(a) Differentiating and using (∗): as

$$\partial(d_1 - d_2)/\partial t = \partial(\sigma\sqrt{T-t})/\partial t = -\frac{1}{2}\sigma/\sqrt{T-t}:$$

$$\Theta = \partial c_t/\partial t = S\phi(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial t}:$$

$$\Theta = Ke^{-r(T-t)}[\phi(d_2)\frac{\partial(d_1-d_2)}{\partial t} - r\Phi(d_2)] = -Ke^{-r(T-t)}[\phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}} + r\Phi(d_2)]:$$

$$\Theta < 0. \qquad \textbf{[8]}$$

(b) Interpretation: an option is (partly) an insurance against future uncertainty. As time passes, there is less future (till expiry) to protect against, so such protection becomes less valuable. **[5]**

(ii) *Puts*. Given the corresponding BS formula for European puts,

$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1),$$

(a) As above, as $\phi(-x) = \phi(x)$,

$$\Theta = \partial p_t/\partial t = rKe^{-r(T-t)}\Phi(-d_2) + Ke^{-r(T-t)}\phi(d_2)\frac{\partial(-d_2)}{\partial t} - S\phi(d_1)\frac{\partial(-d_1)}{\partial t}:$$

$$\Theta = Ke^{-r(T-t)}[r\Phi(-d_2) + \phi(d_2)\frac{\partial(d_1-d_2)}{\partial t}] = Ke^{-r(T-t)}[r\Phi(-d_2) - \phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}}].$$

This can change sign! **[7]**

(b) The situation with puts is different, because of the different role of the strike K (fixed, while S varies). But for large enough K (when a put option – the right to *sell* at price K – will be deeply in the money), the option stands to make a large profit – so the nearer this is to being realised, the better. **[5]**

[Seen – Test 2]

Q4 (*Vasicek model: Objective and risk-neutral measures; statistics, historical estimation*).
(i) Write the objective (real-world) probability measure as $\mathbb{P} = \mathbb{Q}^0$ and the risk-neutral measure as $\mathbb{P}^* = \mathbb{Q}$. The dynamics of the Vasicek model for r under these two measures are:

$$dr_t = [\kappa\theta - (\kappa + \lambda\sigma)r_t]dt + \sigma dW_t^0, \quad r(0) = r_0, \quad \text{under } \mathbb{Q}^0, \qquad (Obj)$$

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t, \quad r(0) = r_0, \quad \text{under } \mathbb{Q}; \qquad (Risk - N)$$

here the parameter λ is the *Sharpe index* ('market price of risk'). Both are expressed by linear Gaussian SDEs, which coincide for $\lambda = 0$.

Pass between these two dynamics by a *Girsanov change of measure*:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^0}|\mathcal{F}_t = \exp\{-\frac{1}{2}\int_0^t \lambda^2 r_s^2 ds + \int_0^t \lambda r_s dW_s^0\}.$$

We get a spot-rate process which is tractable under both measures. **[10]**
(ii) In *modelling*, we begin with the objective measure and dynamics under $\mathbb{Q}^0$, and then pass to the risk-neutral measure and dynamics under $\mathbb{Q}$. In *pricing* one goes in the reverse direction: when pricing, one starts with the risk-neutral dynamics. **[5]**
(iii) *Statistics, historical estimation, econometrics: objective measure*:
We need tractable dynamics in $(RiskN)$: claims are *priced* that way, and we have to be able to price things if we are going to trade in them. We also need tractable dynamics in (Obj) to be able to do statistics on past data.

Say we are given a series $r_0, r_1, \cdots, r_n$ of daily observations of a proxy of r_t – say, a monthly rate, $r_t \sim L(t, t+1m)$: to use this information in our model, we estimate the model parameters. Now data are collected in the *real world*, under the real-world – objective – measure $\mathbb{Q}^0$. So what we can estimate from such historical observations is the $\mathbb{Q}^0$-dynamics, via estimates of the objective parameters κ, λ, θ, σ, by *maximum-likelihood estimation* (MLE). But to price derivatives, we use the risk-neutral measure $\mathbb{Q}$. So calibration of the model to derivative prices, reflecting the current market prices of such derivatives, involves the $\mathbb{Q}$-dynamics, and the parameters κ, θ, σ – but *not* λ, which needs to be estimated from historical data.

As σ is the same in both, we can estimate σ from past data by MLE, and κ, θ by calibration to market prices. **[10]**
[Seen – lectures]

Q5 (*The Heath-Jarrow-Morton (HJM) model and drift condition*).
(i) The forward LIBOR rate at time t between T and S ($S > T > t$),

$$F(t,T,S) = \Big(\frac{P(t,T)}{P(t,S)} - 1\Big)/(T-S),$$

makes the forward-rate agreement (FRA) contract to lock in at time t the interest rates between T and S fair. When S collapses to T, we get the *instantaneous forward rates*, $f(t,T)$. As

$$P(t,T+\delta T) \sim P(t,T) + \delta T \frac{\partial P(t,T)}{\partial T} = P(t,T)[1 + \delta T \frac{\partial P(t,T)}{P(t,T)}]$$

$$= P(t,T)[1 + \delta T \frac{\partial \log P(t,T)}{\partial T}],$$

this gives

$$f(t,T) = \lim_{S \downarrow T} F(t,T,S) \sim \frac{\partial}{\partial T} \log P(t,T).$$

Here we model the *forward rate* $f(t,T)$. **[7]**
(ii) Heath, Jarrow and Morton (1992) – HJM – assumed that, for a given maturity T, the instantaneous forward rate $f(t,T)$ evolves, under a given measure, according to the following diffusion process:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t,$$

with initial condition

$$f(0,T) = f^M(0,T),$$

where

$$T \mapsto f^M(0,T)$$

is the market instantaneous forward curve at time $t = 0$, and $W = (W_1, \cdots, W_N)$ is an N-dimensional BM. Here $\sigma(t,T) = (\sigma_1(t,T), \cdots, \sigma_N(t,T))$ and $\alpha(t,T)$ are adapted processes, and

$$\sigma(t,T)dW_t = \sum_1^N \sigma_i(t,T)dW_i(t)$$

is the dot (scalar) product of the two vectors on the LHS. **[7]**
(iii) The fundamental result of HJM is that, *if the model has no arbitrage* (is

NA), then under the risk-neutral measure the dynamics of f must be of the form

$$df(t,T) = \sigma(t,T)\Big(\int_t^T \sigma(t,s)ds\Big)dt + \sigma(t,T)dW_t. \qquad (HJM)$$

As the coefficient of dt is the (local) mean or drift, and this shows that *the drift is determined by the (local) volatility or diffusion coefficient.* **[7]**

(iv) The SDE (HJM) is called the *Heath-Jarrow-Morton drift condition.* Its best proof involves the Change-of-Numeraire Formula. **[4]**

[Seen – lectures]

Q6 (*Defaultable bonds; Lando's formula.*

A strictly positive stochastic process $t \mapsto \lambda_t$, called the *default intensity* or *hazard rate*, is given for the bond issuer or the CDS reference name. The *cumulative intensity* or *hazard function* is the integrated process

$$\Lambda : \quad t \mapsto \Lambda_t := \int_0^t \lambda_s ds.$$

The *default time* τ can then be defined as the inverse of the process Λ applied to an exponentially distributed ξ with mean 1 and independent of λ:

$$\xi \sim E(1) : \qquad \mathbb{Q}(\xi > u) = e^{-u}, \quad \mathbb{Q}(\xi < u) = 1 - e^{-u}, \quad E[\xi] = 1,$$

$$\tau = \Lambda^{-1}(\xi), \qquad \xi = \Lambda(\tau) \sim E(1), \quad \text{independent of } \lambda. \qquad [\mathbf{4}]$$

Now the probability of surviving for time t is

$$\begin{aligned} \mathbb{Q}(\tau > t) &= \mathbb{Q}(\Lambda^{-1}(\xi) > t) = \mathbb{Q}(\xi > \Lambda(t)) = E[I(\xi > \Lambda(t))] \\ &= E[E[I(\xi > \Lambda(t))|\mathcal{F}_t]] \qquad \text{(Conditional Mean Formula)} \\ &= E[e^{-\Lambda(t)}] \qquad (\xi \sim E(1)) \\ &= E[\exp\{-\int_0^t \lambda_s ds\}] \qquad [\mathbf{4}] \end{aligned}$$

– the bond price if we replace r by λ! Recall that for non-defaultable bonds,

$$P(t,T) = E_t[\frac{B_t}{B_T} 1] = E_t[\exp(-\int_t^T r_s ds)] = E_t[D(t,T)]. \qquad (P) \qquad [\mathbf{3}]$$

Theorem (Lando's formula). The price of a defaultable bond is the price of a default-free bond, *with the risk-free short-rate r replaced by $r + \lambda$.* [**4**]

Proof.

$$\begin{aligned} \overline{P}(0,T) &= E[D(0,T)I(\tau > T)] \\ &= E[\exp\{-\int_0^T r_s ds\} I(\Lambda^{-1}(\xi) > T)] \\ &= E[\exp\{-\int_0^T r_s ds\} I(\xi > \Lambda(T))] \\ &= E[E[\exp\{-\int_0^T r_s ds\} I(\xi > \Lambda(T))|\Lambda, r]] \qquad \text{(Tower property)} \end{aligned}$$

$$
\begin{aligned}
&= E[\exp\{-\int_0^T r_s ds\}]E[I(\xi > \Lambda(T))|\Lambda] \qquad \text{(independence)} \\
&= E[\exp\{-\int_0^T r_s ds\}]\ \mathbb{Q}(\xi > \Lambda(T))|\Lambda) \qquad (E[I(.) = \mathbb{Q}(.)) \\
&= E[\exp\{-\int_0^T r_s ds\} \exp\{-\Lambda(T)\}] \qquad \text{(by above)} \\
&= E[\exp\{-\int_0^T r_s ds\} \exp\{-\int_0^T \lambda_s ds\}] \qquad \text{(def. of } \Lambda) \\
&= E[\exp\{-\int_0^T (r_s + \lambda_s) ds\}]. \qquad // \qquad \mathbf{[10]}
\end{aligned}
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[Seen – lectures]