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MATL481 INTEREST RATE THEORY: MOCK EXAM SOLUTIONS 2017

Q1. *The business cycle; the Crash; quantitative easing (QE); persistent depression*

The Business Cycle

The traditional view here is that when the economy was expanding – 'boom', with demand and activity increasing – firms would compete for labour, wages would rise, costs would rise, prices would rise, inflation would rise. The central bank would *increase* interest rates – Bank rate – to make borrowing money more expensive. This would decrease the demand for borrowing by business, and the economy would contract. By contrast, when the economy was contracting – 'bust', or 'slump' – the Bank would reduce interest rates, to make it cheaper for business to borrow. This would have the effect of making business expansion cheaper; businesses would tend to expand. The expansion would tend to overshoot the natural mean position, leading to the next expansion and the next business cycle. **[7]**

The Crash; quantitative easing (QE).

Since the Crash of 2007/08 the economy has been consistently flat. In an effort to promote growth, the authorities have held interest rates at historically low levels for long periods. In the UK, bank rate is now 0.25%, down from 0.5%, itself unprecedentedly low. The authorities have also resorted to unconventional monetary measures, such as *quantitative easing (QE)*, usually described informally as creating electronic money. This has had the desired effect of moving the economy back towards normal, from the crisis of the Crash and its immediate aftermath. But, QE has had undesirable and unpredicted effects. In particular, it has led to a large increase in asset prices. This had benefited those who hold assets – principally, the already affluent. This has widened the gap between the rich and the poor, decreasing social mobility and increasing social and political tensions. In addition, low interest rates have penalised savers. This is both unfair to them, and undesirable nationally: we suffer from an excess of consumer indebtedness, so saving should be encouraged. **[7]**

Persistent depression

The major western economies have been very slow to recover from the Crash of 07/08. This is not unprecedented: the Japanese economy has had

similar – and worse – experiences. After the devastation of WWII, and American occupation, the Japanese economy experienced an 'economic miracle', similar to that in Germany. From the late 50s to around 1990, Japan had a dominant position in several areas of manufacturing: ship-building (oil tankers and super-tankers), steel, cars, electronics (from transistor radios on), etc. There was then a financial crisis – perhaps a precursor of the western Crash of 2007/08, which involved an asset-price bubble – bubbles burst! The economy was stagnant throughout the 90s, described as Japan's lost decade. But things have been little better since (lost decade*s*). **[6]**
[Seen – lectures]

Q2. *Prelude to Black's caplet formula*

$$dF(t;T_1,T_2) = \sigma_2(t)F(t;T_1,T_2)dW_2(t), \qquad \text{IC } mkt\ F(0;T_1,T_2), \quad (LMM)$$

To solve the SDE (LMM) above, and compute $E_2[\tau(F_2(T_1)-K)_+]$: by Itô's formula, as $\log' x = 1/x$, $\log'' x = -1/x^2$, $(dW_2(t))^2 = dt$, (LMM) gives

$$\begin{aligned} d\log F_2(t) &= \frac{1}{F_2}dF_2 + \frac{1}{2}(-\frac{1}{F_2^2})dF_2 dF_2 \\ &= \frac{1}{F_2}\sigma_2 F_2 dW_2 + \frac{1}{2}(-\frac{1}{F_2^2}(\sigma_2 F_2 dW_2)^2 \\ &= \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2 dt : \end{aligned}$$

$$d\log F_2(t) = \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2 dt.$$

Integrate both sides:

$$\log F_2(T) - \log F_2(0) = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt :$$

$$F_2(T) = F_2(0)\exp\{\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt\}.$$

The distribution of the random variable in the exponent is Gaussian, since it is a stochastic integral of a deterministic function by a Brownian motion (MATL480 Problems 5b Q1 – sums of independent Gaussians is Gaussian). Compute its expectation: as the Itô integral has mean 0,

$$E[\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt] = -\frac{1}{2}\int_0^T \sigma_2(t)^2 dt.$$

The variance is

$$var(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2 dt) = var(\int_0^T \sigma_2(t)dW_2(t))$$

(as the second term is deterministic)

$$= E[(\int_0^T \sigma_2(t)dW_2(t))^2] \qquad \text{(as the mean is 0)}$$

$$= \int_0^T \sigma_2(t)^2 dt, \qquad \text{(by Itô's isometry: MATL480, V.5).}$$

Summarising,

$$I(T) := \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2(t)^2 dt \sim m + VN(0,1)$$

(here '$\sim m + VN(0,1)$' is shorthand for 'is distributed as $m + V$ times a $N(0,1)$ – a standard normal random variable'), where

$$m = -\frac{1}{2} \int_0^T \sigma_2(t)^2 dt, \qquad V^2 = \int_0^T \sigma_2(t)^2 dt.$$

That is,

$$F_2(T) = F_2(0) \exp\{I(T)\} = F_2(0) e^{m+VZ}, \qquad Z \sim N(0,1).$$

Schoenmakers-Coffey parametrisation of correlations

Schoenmakers and Coffey propose a finite sequence

$$1 = c_1 < c_2 < \cdots < c_M, \qquad c_1/c_2 < c_2/c_3 < \cdots < c_{M-1}/c_M,$$

and they set (F here stands for Full (Rank))

$$\rho^F(c)_{ij} := c_i/c_j, \qquad i \le j, \qquad i, j = 1, \cdots, M. \tag{SC}$$

So the correlation between changes in adjacent rates is

$$\rho^F_{i+1,i} = c_i/c_{i+1};$$

these are all < 1, and are *increasing* in i. Both these are desirable features, in view of the above.

So: under (SC), *the subdiagonal of the correlation matrix* $\rho^F(c)$ *is increasing when moving from NW to SE.* Interpretation: as we move along the yield curve, the larger the tenor, the more correlated changes in adjacent forward rates become. This corresponds (not only to the expectation above, but also) to the experienced fact that the forward curve tends to flatten, and to move in a more correlated way, for large maturities than for small ones.

The number of parameters needed for a Schoenmakers-Coffey matrix is M, rather than the $\frac{1}{2}M(M-1)$ parameters needed for a general correlation matrix of the same size. One can show (we quote this) that any such SC matrix is a genuine correlation matrix – symmetric, positive semi-definite and with 1s on the diagonal.

Schoenmakers and Coffey also observed that this parametrisation can be characterised alternatively in terms of

$$\Delta_2, \cdots, \Delta_M \ge 0:$$

$$c_i = \exp\{\sum_{j=2}^{i} j\Delta_j + \sum_{j=i+1}^{M} (i-1)\Delta_j\}. \tag{SCΔ}$$

Full-rank, two-parameter, exponentially decreasing parametrisation

Schoenmakers and Coffey also introduced the model

$$\rho_{i,j} = \rho_\infty + (1-\rho_\infty)\exp\{-\beta|i-j|\}, \qquad \beta \ge 0.$$

Here ρ_∞ still represents the correlation between the ends, but only asymptotically (let $j \to \infty$).

Q4. *Dupire's formula*

Suppose we have an option on the forward rate $F(T)$, with payoff function h and expiry T. For $t \in [0,T]$, if

$$v(t,x) := E[h(F_T)|F_t = x],$$

$$\begin{aligned} E[h(F_T)] &= E[E[h(F_T)|F_t = x]] \qquad \text{(tower property)} \\ &= \int_0^\infty v(t,x)\phi(t,x)dx, \end{aligned}$$

if F_t has density $\phi(t,x)$. Now the LHS is *independent* of t. Hence, so too is the RHS: differentiating under the integral sign w.r.t. t as above,

$$0 = \int \frac{\partial v}{\partial t}\phi dx + \int v \frac{\partial \phi}{\partial t} dx.$$

Now, v satisfies the*Kolmogorov backward equation* (*Fokker-Planck equation*):

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma(t,x)^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \qquad v(T,x) = h(x). \qquad (FoPl)$$

By $(FoPl)$, we can substitute for the $\partial v/\partial t$ term in the above, to obtain (writing v' for $\partial v/\partial x$, etc.)

$$0 = -\frac{1}{2}\int (\sigma^2 x^2 \phi) v'' dx + \int v \frac{\partial \phi}{\partial t} dx. \qquad (*)$$

Integrate the first integral by parts: the integrated term vanishes (at 0 because of the x^2, at infinity because the other factors decay fast enough):

$$\int (\sigma^2 x^2 \phi) v'' dx = \int (\sigma^2 x^2 \phi) dv' = -\int (\sigma^2 x^2 \phi)' v' dx = -\int (\sigma^2 x^2 \phi)' dv.$$

Integrate by parts again: again the integrated terms vanish, giving

$$\int (\sigma^2 x^2 \phi) dv = \int v (\sigma^2 x^2 \phi)'' dx.$$

Substituting this in $(*)$,

$$0 = \int (\frac{1}{2}(\sigma^2 x^2 \phi - \frac{\partial \phi}{\partial \tau}) v dx.$$

But the payoff h, and so the conditional density v, is arbitrary. So the integrand here must vanish, giving the *forward equation* (as it deals with forward rates),

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma(t,x)^2 x^2 \phi). \qquad (ForEq)$$

Suppose now that the option above is a call C with strike K. Then

$$C(T,K) = E[(F-K)_+] = E[(F-K)I(F>K)] = \int_K^\infty (x-K)\phi(t,x)dx.$$

So, first differentiating under the integral sign w.r.t. K,

$$\partial C(T,K)/\partial K = -\int_K^\infty \phi(T,x)dx$$

(the $(x-K)$ term vanishes at the lower limit). So

$$\partial^2 C(T,K)/\partial K^2 = \phi(T,K). \qquad (**)$$

Next, differentiate w.r.t. T under the integral sign and use $(ForEq)$:

$$\begin{aligned}\frac{\partial C(T,K)}{\partial T} &= \int_K^\infty (x-K)\frac{\partial \phi(T,x)}{\partial T}dx \\ &= \int_K^\infty (x-K).\frac{1}{2}(\sigma^2 x^2 \phi)'' dx \qquad \text{(by } (ForEq)) \\ &= -\frac{1}{2}\int_K^\infty (\sigma^2 x^2\phi)' dx = -\frac{1}{2}\int_K^\infty d(\sigma^2 x^2 \phi) \qquad \text{(integrating by parts)} \\ &= \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K) \qquad \text{(lower limit, hence the -)},\end{aligned}$$

performing the integration. This gives, by $(**)$:

Theorem (Dupire's formula). In the notation above, the call price satisfies

$$C(T,K) = \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K).$$

That is, the *local volatility* $\sigma(T,K)$ is completely specified by the *volatility surface* $\sigma(K,T)$ (via its derivatives) by Dupire's formula,

$$\sigma(T,K) = \frac{1}{K}\sqrt{\frac{2\partial C(T,K)/\partial T}{\partial^2 C(T,K)/\partial K^2}}. \qquad (Dup)$$

Q5. *Defaultable bonds; Lando's formula*

A strictly positive stochastic process $t \mapsto \lambda_t$, called the *default intensity* or *hazard rate*, is given for the bond issuer or the CDS reference name. The *cumulative intensity* or *hazard function* is the integrated process

$$\Lambda : \quad t \mapsto \Lambda_t := \int_0^t \lambda_s ds.$$

The *default time* τ can then be defined as the inverse of the process Λ applied to an exponentially distributed ξ with mean 1 and independent of λ:

$$\xi \sim E(1) : \qquad \mathbb{Q}(\xi > u) = e^{-u}, \quad \mathbb{Q}(\xi < u) = 1 - e^{-u}, \quad E[\xi] = 1,$$

$$\tau = \Lambda^{-1}(\xi), \qquad \xi = \Lambda(\tau) \sim E(1), \quad \text{independent of } \lambda. \qquad [\mathbf{3}]$$

Now the probability of surviving for time t is

$$\begin{aligned} \mathbb{Q}(\tau > t) &= \mathbb{Q}(\Lambda^{-1}(\xi) > t) = \mathbb{Q}(\xi > \Lambda(t)) = E[I(\xi > \Lambda(t))] \\ &= E[E[I(\xi > \Lambda(t))|\mathcal{F}_t]] \qquad \text{(Conditional Mean Formula)} \\ &= E[e^{-\Lambda(t)}] \qquad (\xi \sim E(1)) \\ &= E[\exp\{-\int_0^t \lambda_s ds\}] \qquad [\mathbf{3}] \end{aligned}$$

– the bond price if we replace r by λ! Recall that for non-defaultable bonds,

$$P(t,T) = E_t[\frac{B_t}{B_T} 1] = E_t[\exp(-\int_t^T r_s ds)] = E_t[D(t,T)]. \qquad (P) \qquad [\mathbf{1}]$$

Theorem (Lando's formula). The price of a defaultable bond is the price of a default-free bond, *with the risk-free short-rate r replaced by $r + \lambda$.* [**3**]

Proof.

$$\begin{aligned} \overline{P}(0,T) &= E[D(0,T)I(\tau > T)] \\ &= E[\exp\{-\int_0^T r_s ds\} I(\Lambda^{-1}(\xi) > T)] \\ &= E[\exp\{-\int_0^T r_s ds\} I(\xi > \Lambda(T))] \\ &= E[E[\exp\{-\int_0^T r_s ds\} I(\xi > \Lambda(T))|\Lambda, r]] \qquad \text{(Tower property)} \end{aligned}$$

$$
\begin{aligned}
&= E[\exp\{-\int_0^T r_s ds\}] E[I(\xi > \Lambda(T))|\Lambda] \qquad \text{(independence)} \\
&= E[\exp\{-\int_0^T r_s ds\}] \ \mathbb{Q}(\xi > \Lambda(T))|\Lambda) \\
&= E[\exp\{-\int_0^T r_s ds\} \exp\{-\Lambda(T)\}] \\
&= E[\exp\{-\int_0^T r_s ds\} \exp\{-\int_0^T \lambda_s ds\}] \\
&= E[\exp\{-\int_0^T (r_s + \lambda_s) ds\}]. \qquad // \qquad \mathbf{[10]}
\end{aligned}
$$

[Seen – lectures]