MATL481 INTEREST RATE THEORY: RESIT EXAM 2017-18

Six questions; do four; 25 marks per question

Q1. Comment briefly on:
(i) collaterized debt obligations (CDOs);
(ii) toxic debt;
(iii) securitization;
(iv) negative interest rates.

Q2. Define: bond prices $P(t,T)$; spot rates r_t; forward rates $f(t,T)$. Describe the relationships between them. Give examples of each. Which of them can we measure?

Q3. Recall the Black-Scholes formula for the price c_t of European calls,

$$c_t = S_t\Phi(d_1) - Ke^{-r\tau}\Phi(d_2),$$

with S_t the stock price at time $t \in [0,T]$, K the strike price, r the riskless interest rate, σ the volatility, $\tau := T - t$,

$$d_1 := [\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}, \qquad d_2 := d_1 - \sigma\sqrt{\tau},$$

and put-call parity $S + p - c = Ke^{-r\tau}$.
(i) Show that

$$Ke^{-r\tau}\phi(d_2) = S\phi(d_1),$$

with ϕ the standard normal density function.
(ii) Show that for calls c, $\rho := \partial c/\partial r > 0$.
(iii) Give the financial interpretation of (ii).
(iv) Show that for puts p, $\rho := \partial p/\partial r < 0$.
(v) Give the financial interpretation of (iv).
(vi) Do these results extend to American options? If so, why?

Q4. Given the stochastic differential equation for the LIBOR market model,

$$dF(t;T_1,T_2) = \sigma_2(t)F(t;T_1,T_2)dW_2(t), \qquad (LMM)$$

show that

$$F_2(T_1) = F_2(0)e^{m+VZ}, \qquad Z \sim N(0,1),$$

where

$$m = -\frac{1}{2}\int_0^{T_1} \sigma_2(t)^2 dt, \qquad V^2 = \int_0^{T_1} \sigma_2(t)^2 dt.$$

Q5. The Ornstein-Uhlenbeck SDE is

$$dV_t = -\kappa V dt + \sigma dW_t, \qquad V_0 = v_o, \qquad (OU)$$

with W Brownian motion.
(i) Give the interpretation of the parameters κ, σ.
(ii) Solve (OU) to obtain

$$V = v_0 e^{-\kappa t} + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} dW_s.$$

(iii) Show that V_t is Gaussian, with mean $v_o e^{-\kappa t}$ and variance

$$Var(V_t) = \frac{\sigma^2 [1 - e^{-2\kappa t}]}{2\kappa}.$$

(iv) Show that the covariance is

$$cov(V_t, V_{t+u}) = \frac{\sigma^2 e^{-\kappa u}[1 - e^{-2\kappa t}]}{2\kappa}.$$

(v) Show that the process V is Markov, and has a limit at $t \to \infty$ which is stationary Gaussian Markov.
(vi) What is meant by saying that V is *mean-reverting*? What is the link with the Vasicek model for the short rate of interest?
(vii) Comment briefly on the limitations of the model.

Q6. Given that

$$v(t, x) := E[h(F_T) | F_t = x],$$

satisfies the Fokker-Planck equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma(t, x)^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \qquad v(T, x) = h(x), \qquad (FoPl)$$

and writing $\phi(t, x)$ for the density of F_t, show that, for h a call C with strike K,

$$\partial^2 C(T, K)/\partial K^2 = \phi(T, K).$$

Hence or otherwise show that:
(i) the call $C(T, K)$ and density $\partial(T, K)$ are linked by

$$\partial C(T,K)/\partial T = \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K);$$

(ii) the *local volatility* $\sigma(T, K)$ is completely specified by the call-price surface $C(.,.)$ through its derivatives by Dupire's formula,

$$\sigma(T,K) = \frac{1}{K}\sqrt{\frac{2\partial C(T,K)/\partial T}{\partial^2 C(T,K)/\partial K^2}}. \qquad (Dup)$$

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