MATL481 INTEREST RATE THEORY: RESIT EXAM SOLUTIONS 2017

Q1: ZCB; Libor; Bond price/Libor; QE
(i) Zero-Coupon Bonds (ZCB)

A *T-maturity zero-coupon bond (ZCB)* is a contract which guarantees the payment of one unit of currency at time T. The contract value at time $t \in [0, T]$ is denoted by $P(t, T)$. So $P(T, T) = 1$, and writing $D = D(t, T)$ for the *discount function* $D(t, T) := \exp(-\int_t^T r_s ds)$, with r_t the *spot-rate* at time t (instantaneous riskless interest rate), the price (value) at t is (expectation under $\mathbb{Q}$-measure (risk-neutral measure)

$$P(t, T) = E_t[\frac{B_t}{B_T}1] = E_t[\exp(-\int_t^T r_s ds)] = E_t[D(t, T)]. \qquad (P) \ [\mathbf{5}]$$

(ii) Libor

The *spot-LIBOR rate* $L(t, T)$ at time t for maturity T is the constant rate at which an investment has to be made to produce an amount of one unit of currency at maturity, starting from $P(t, T)$ units of currency at time t, when accruing occurs *proportionally* to the investment time:

$$P(t, T)(1 + (T - t)L(t, T)) = 1, \qquad L(t, T) = \frac{1 - P(t, T)}{(T - t)P(t, T)}. \ (P - L) \ [\mathbf{5}]$$

(iii) Bond price/LIBOR.

From $(P - L)$: when bond prices P go up, Libor rates L go down; when bond prices go down, Libor goes up: the two are *inversely related.* **[5]**

(iv) Quantitative easing (QE)

This is a process whereby the Government buys its own bonds (gilts). This increases the bond price (ZCB) (by supply and demand), so decreases Libor (interest rates). This decreases interest rates generally, and so Bank rate, and has been used since the Crash of 2007/8 to encourage economic activity. **[5]**

[Mainly seen, lectures]

Q2: Afffine term-structure models
(i) Affine term-structure models (ATM for short) are those for which the continuously compounded spot rate $R(t,T)$ (II.1) is an *affine function of the spot rate* r_t:

$$R(t,T) = \alpha(t,T) + \beta(t,T)r_t, \qquad (ATM)$$

or in terms of the bond price $P(t,T)$,

$$P(t,T) = A(t,T)\exp\{-B(t,T)r_t\}. \qquad [\mathbf{4}]$$

(ii) In terms of the *instantaneous forward rate*

$$f(t,T) := -\frac{\partial}{\partial T}\log P(t,T), \qquad P(t,T) = \exp\{-\int_t^T f(t,u)du\}.$$

So for affine models,

$$f(t,T) = -\frac{\partial}{\partial T}\log A(t,T) + \frac{\partial B(t,T)}{\partial T}r_t. \qquad [\mathbf{4}]$$

(iii) So the stochastic differential is of the form

$$df(t,T) = \{\cdots\}dt + \frac{\partial B(t,T)}{\partial T}\sigma(t,r_t)dW_t,$$

where $\sigma(t,r_t)$ is the diffusion coefficient in the short-rate dynamics for r. So the volatility for f in an affine model is

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t,r_t). \qquad [\mathbf{4}]$$

(iv) Write the risk-neutral dynamics for the short rate r_t as

$$dr_t = b(t,r_t)dt + \sigma(t,r_t)dW_t.$$

If both the functions b and σ^2 are *affine themselves*:

$$b(t,x) = \lambda(t)x + \eta(t), \qquad \sigma^2(t,x) = \gamma(t)x + \delta(t),$$

then the functions A and B can be obtained from the functions λ, η, γ, δ above by solving the following differential equations (DEs):

$$\frac{\partial}{\partial t}B(t,T) + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 + 1 = 0, \qquad B(T,T) = 1,$$

$$\frac{\partial}{\partial t}\log A(t,T) - \eta(t)B(t,T) + \frac{1}{2}\delta(t)B(t,T)^2 = 0, \qquad A(T,T) = 1. \qquad [\mathbf{4}]$$

(v) For the Vasicek model,

$$\lambda(t) = -\kappa, \qquad \eta(t) = \kappa\theta, \qquad \gamma(t) = 0, \qquad \delta(t) = \sigma^2. \qquad [\mathbf{4}]$$

[Seen, lectures]

Q3: Heath-Jarrow-Morton (HJM)
(i) The Heath, Jarrow and Morton (HJM) model of 1992 assumes that, for a given maturity T, the instantaneous forward rate $f(t,T)$ evolves, under a given measure, according to the following diffusion process:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \qquad f(0,T) = f^M(0,T),$$

where

$$T \mapsto f^M(0,T)$$

is the market instantaneous forward curve at time $t=0$, and $W = (W_1, \cdots, W_N)$ is an N-dimensional BM. Here $\sigma(t,T) = (\sigma_1(t,T), \cdots, \sigma_N(t,T))$ and $\alpha(t,T)$ are adapted processes, and

$$\sigma(t,T)dW_t = \sum_1^N \sigma_i(t,T)dW_i(t)$$

is the dot (scalar) product of the two vectors on the LHS. **[5]**
(ii) The HJM drift condition.

The fundamental result of HJM is that, *if the model has no arbitrage* (is NA), then under the risk-neutral measure the dynamics of f must be of the form (*Heath-Jarrow-Morton drift condition*)

$$df(t,T) = \sigma(t,T)\Big(\int_t^T \sigma(t,s)ds\Big)dt + \sigma(t,T)dW_t. \qquad (HJM)$$

As the coefficient of dt is the (local) mean or drift, and this shows that:
the drift is determined by the (local) volatility or diffusion coefficient. **[5]**
(iii) Gaussian HJM model.

If the volatility σ in (HJM) is deterministic, then the forward rate f satisfies in (HJM) an SDE of Itô type. We can write this as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad \alpha(t,T) = \sigma(t,\sigma(t,T)\Big(\int_t^T \sigma(t,s)ds\Big)dt. \qquad (G-HJM)$$

This has solution

$$f(t,T) = f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T).dW_u,$$

which is Gaussian (as Gaussianity is preserved under linear operations, such as integration). **[5]**

(iv) As the spot rate r_t satisfies

$$f(t,T) = E[r_T|\mathcal{F}_t] \qquad (t \in [0,T])$$

(expectation under the risk-neutral measure), so

$$r_t = f(t,t),$$

this too is Gaussian, taking $T = t$ above. **[5]**

[Seen – lectures]

Q4. Market models; Schoenmakers-Coffey matrices

(i) Market models.

In a nutshell: *Don't try to model infinite-dimensional things you can't see. Model instead finite-dimensional things you can see.*

What makes this work is that, although interest rates are in principle *infinite-dimensional* – the yield curve, or term-structure of interest rates, is an infinite-dimensional object – because only finitely many products are traded in the market (which ones are determined by the tenor structure), and these are highly liquid, all we really need is to model these. In practice, this largely reduces to modelling two things: the *correlations*, and the *volatilities*. **[5]**

(ii) Market models: Impact

Before market models were introduced (in 1997), short-rate models were the main choice for pricing and hedging of interest-rate derivatives. They are still used for many applications, and are based on modelling the instantaneous short rate – spot rate – r_t via a (perhaps multidimensional) diffusion process. This diffusion characterises the evolution of the complete yield curve in time. Short-rate models were followed by forward-rate models.

It is better to model what one can actually *see*. This is the prices at which liquid products are traded, in the *market*. *This is what market models do*. One cannot actually *see* forward rates and short rates. **[5]**

(iii) Schoenmakers-Coffey (S&C) matrices.

Schoenmakers and Coffey propose a finite sequence

$$1 = c_1 < c_2 < \cdots < c_M, \qquad c_1/c_2 < c_2/c_3 < \cdots < c_{M-1}/c_M,$$

and they set (F here stands for Full (Rank))

$$\rho^F(c)_{ij} := c_i/c_j, \qquad i \le j, \qquad i, j = 1, \cdots, M. \qquad (SC)$$

They showed (by Linear Algebra) that such a matrix $C = (\rho^F(c)_{ij})_{ij}$ is indeed a correlation matrix (i.e. is *positive definite*). **[5]**

(iv) Behaviour for large maturities

From (SC), the correlation between changes in adjacent rates is

$$\rho^F_{i+1,i} = c_i/c_{i+1};$$

these are all < 1, and are *increasing* in i. This is a realistic feature: as we move along the yield curve, the larger the tenor, the more correlated changes

in adjacent forward rates become. For, the further we go into the future, the less we can discriminate between what happens at time-points a fixed distance apart. So the forward curve tends to flatten, and to move in a more correlated way, for large maturities than for small ones. **[5]**
[Seen – lectures]

Q5: CDOs; toxic debt; securitisation; negative interest rates

(i) Collateralised debt obligations (CDOs)

A CDO is a structured financial product that pools together cash-flow-generating assets (mortgages, bonds, loans etc.), and repackages this asset pool into discrete *tranches*, that can be sold to investors. The senior tranches have priority – get repaid first – in case of default; they thus have higher credit ratings, but offer lower coupon rates. Conversely, the junior tranches have lower credit ratings, but offer higher coupon rates to compensate for this.

CDOs split, into *mortgage-backed securities (MBS)*, and *asset-backed securities (ABS).* **[5]**

(ii) Toxic debt

Many of the CDOs that banks owned were based on assets in the sub-prime mortgage area. When the sub-prime bubble burst, the value of such CDOs burst with it – with devastating consequences: the Crash. It emerged that the boards of the big banks did not understand the dangers they had been running. They did not know what their CDOs and other such assets were worth. It was a great shock to banks to realise that they had no idea what their assets were worth. Worse: they realised that other banks were in the same situation. The result was a sudden collapse in the confidence of banks *in both themselves and other banks.* So banks abruptly stopped lending – even *to each other.* When the inter-bank lending that provides the *lubrication* that keeps the wheels of finance turning was withdrawn, the wheels stopped turning and the economy seized up. **[5]**

(iii) Securitization

Securitization is the name given to the search in recent decades for new opportunities for profit, based on identifying risks that people or firms will want protection from (or insurance against). Of course, taking risks is risky: it could go wrong. But, 'nothing venture, nothing win': businesses know that they cannot make profits without engaging in market activity, and this is risky. Business (at least in some sectors – investment banking, for example) has an appetite for risk, for this reason. As a result, there are now all kinds of (fairly) new derivatives: weather derivatives; catastrophe derivatives ('cat bonds'); volatility derivatives (VIX index), etc.

Recall the role of catastrophes such as major US hurricanes, the wave of asbestos claims etc. in the Lloyds of London insurance scandal of the 1990s, and what it revealed about the lack of proper oversight (within Lloyds), and regulation (outside it). **[5]**

(iv) Negative interest rates

(iv) Negative interest rates

Interest rates have always been regarded as naturally positive, as they compensate the lender for the two disadvantages of lending money: the risk of default, and the loss (for the loan period) of the use of one's own money. Negative interest rates would have been regarded as ridiculous before the Crash. But, at individual level, banks provide a service in looking after customers' money: protection against theft (or robbery, as was once common), accidental loss etc., and this service could in principle be charged for.

After the Crash, at government/central bank level, interest rates have been held at historically very low rates (fractions of a percent) for extended periods (a decade now). Negative interest rates have indeed been seen, in several major countries. Central banks are thus charging banks for the service of looking after their money, and are encouraging them to lend funds (often publicly provided), to stimulate the economy, rather than hoard them (to shore up their capital reserves), by directly penalising them if they do not do so. **[5]**

[Mainly seen – lectures] NHB