MATL481 INTEREST RATE THEORY: RESIT EXAM SOLUTIONS 2017-18

Six questions; do four; twenty-five marks per question

Q1. *CDOs; toxic debt; securitisation; negative interest rates*
(i) Collateralised debt obligations (CDOs)

A CDO is a structured financial product that pools together cash-flow-generating assets (mortgages, bonds, loans etc.), and repackages this asset pool into discrete *tranches*, that can be sold to investors. The senior tranches have priority – get repaid first – in case of default; they thus have higher credit ratings, but offer lower coupon rates. Conversely, the junior tranches have lower credit ratings, but offer higher coupon rates to compensate for this.

CDOs split, into *mortgage-backed securities (MBS)*, and *asset-backed securities (ABS)*. **[6]**

(ii) Toxic debt

Many of the CDOs that banks owned were based on assets in the sub-prime mortgage area. When the sub-prime bubble burst, the value of such CDOs burst with it – with devastating consequences: the Crash. It emerged that the boards of the big banks did not understand the dangers they had been running. They did not know what their CDOs and other such assets were worth. It was a great shock to banks to realise that they had no idea what their assets were worth. Worse: they realised that other banks were in the same situation. The result was a sudden collapse in the confidence of banks *in both themselves and other banks*. So banks abruptly stopped lending – even *to each other*. When the inter-bank lending that provides the *lubrication* that keeps the wheels of finance turning was withdrawn, the wheels stopped turning and the economy seized up. **[6]**

(iii) Securitization

Securitization is the name given to the search in recent decades for new opportunities for profit, based on identifying risks that people or firms will want protection from (or insurance against). Of course, taking risks is risky: it could go wrong. But, 'nothing venture, nothing win': businesses know that they cannot make profits without engaging in market activity, and this is risky. Business (at least in some sectors – investment banking, for example) has an appetite for risk, for this reason. As a result, there are now all kinds of (fairly) new derivatives: weather derivatives; catastrophe derivatives ('cat bonds'); volatility derivatives (VIX index), etc.

Recall the role of catastrophes such as major US hurricanes, the wave of asbestos claims etc. in the Lloyds of London insurance scandal of the 1990s, and what it revealed about the lack of proper oversight (within Lloyds), and regulation (outside it). **[6]**

(iv) Negative interest rates

Interest rates have always been regarded as naturally positive, as they compensate the lender for the two disadvantages of lending money: the risk of default, and the loss (for the loan period) of the use of one's own money. Negative interest rates would have been regarded as ridiculous before the Crash. But, at individual level, banks provide a service in looking after customers' money: protection against theft (or robbery, as was once common), accidental loss etc., and this service could in principle be charged for.

After the Crash, at government/central bank level, interest rates have been held at historically very low rates (fractions of a percent) for extended periods (a decade now). Negative interest rates have indeed been seen, in several major countries. Central banks are thus charging banks for the service of looking after their money, and are encouraging them to lend funds (often publicly provided), to stimulate the economy, rather than hoard them (to shore up their capital reserves), by directly penalising them if they do not do so. **[7]**

[Mainly seen – lectures]

Q2. *Bond prices; spot rates; forward rates*

Bond prices. The simplest case is that of *zero-coupon bonds*, or ZCBs (*coupons* being payments made to bond-holders analogous to *dividends* being payments made to stock-holders). The price at time $t \in [0, T]$ of a bond paying 1 (unit of currency) at time T (the *maturity*) is $P(t, T)$.

Given what we know 'now', at current time t, bond prices are non-random: ZCBs are highly liquid; we can see them being traded (at various maturities T); so we know what they are worth: we know the $P(t, T)$. **[7]**

Spot rates. Here the (constant, non-random, risk-free) interest rate r of Black-Scholes theory is replaced by a *stochastic process* $r = (r_t)$. They are the instantaneous rates implied by the bond prices $P(t, T)$ above: with $\mathcal{F}_t$ the information available at time t

$$P(t, T) = E_t[\exp\{-\int_t^T r_s ds\}],$$

with $E_t[.] = E_{\mathbb{Q}}[.|\mathcal{F}_t]$ and $\mathbb{Q}$ the risk-neutral measure.

Spot rates are simple conceptually (as they extend the familiar Black-Scholes case so visibly), but they are difficult to measure, as they are *rates*; rates, being instantaneous, are harder to measure than quantities depending on an *interval* of time, such as LIBOR.

Examples: Vasicek model; Cox-Ingersoll-Ross model (CIR); two-factor Vasicek; two-factor CIR. **[9]**

Forward rates. These are the interest rates $f(t, T)$ implied over the time-interval $[t, T]$ by the bond prices $P(t, T)$ above; thus

$$P(t, T) = \exp\{-\int_t^T f(t, s)ds\}.$$

Again, being a *rate*, forward rates cannot be measured directly.

Example: the Heath-Jarrow-Morton (HJM) model, with SDE

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dW_t, \qquad (HJM)$$

with $\alpha(t, T)$ the drift, $\sigma(t, T)$ the volatility and (W_t) Brownian motion.

To avoid arbitrage, the drift α here must be a function of the volatility σ. This is the *Heath-Jarrow-Morton drift condition.* **[9]**

[Seen, lectures]

Q3. *Rho.*
(i) *Rho for calls.*

With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$, $\tau := T - t$ the time to expiry, the Black-Scholes call price is, with d_1, d_2 as given,

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \qquad (BS)$$

So as $d_2 = d_1 - \sigma\sqrt{\tau}$,

$$\phi(d_2) = \frac{e^{-\frac{1}{2}(d_1-\sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*) \quad \mathbf{[4]}$$

(ii) Differentiating (BS) partially w.r.t. r gives, by $(*)$,

$$\begin{aligned} \rho := \partial C/\partial r &= S\phi(d_1)\partial d_1/\partial r - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(d_1 - d_2)/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial r + K\tau e^{-r\tau}\Phi(d_2) = K\tau e^{-r\tau}\Phi(d_2) : \end{aligned}$$

$$\rho > 0. \qquad \mathbf{[4]}$$

(iii) *Financial interpretation.*

As r increases, cash becomes more attractive compared to stock. So stock *buyers* have a 'buyer's market', favouring them. So for *calls* (options to buy), $\rho > 0$. **[4]**

(iv) *Rho for puts.*

By put-call parity, $S + P - C = Ke^{-r\tau}$:

$$\partial P/\partial r = \partial C/\partial r - K\tau e^{-r\tau} = -K\tau e^{-r\tau}[1 - \Phi(d_2)] = -K\tau e^{-r\tau}\Phi(-d_2) < 0. \qquad \mathbf{[4]}$$

(v) *Financial interpretation.*

As above: as r increases, stock *sellers* also operate in a buyer's market, but this is against them. So for *puts* (options to sell), $\rho < 0$. **[4]**

(vi) *American options.*

All this extends to American options,via the *Snell envelope*, which is *order-preserving*. The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As r increases, the Z-terms increase for calls (rho is positive for European calls). As the Zs increase, the Us increase (above: backward induction on n – dynamic programming, as usual for American options). Combining: as r increases, the U-terms increase. So rho is also positive for American calls. Similarly, rho is negative for American puts. **[5]**

[Similar to 'vega positive', done in Problems]

Q4. *LIBOR Market Model (LMM)*

$$dF(t;T_1,T_2) = \sigma_2(t)F(t;T_1,T_2)dW_2(t), \qquad \text{IC } mkt\ F(0;T_1,T_2), \quad (LMM)$$

To solve the SDE (LMM) above, and compute $E_2[\tau(F_2(T_1)-K)_+]$: by Itô's formula, as $\log' x = 1/x$, $\log'' x = -1/x^2$, $(dW_2(t))^2 = dt$, (LMM) gives

$$\begin{aligned} d\log F_2(t) &= \frac{1}{F_2}dF_2 + \frac{1}{2}(-\frac{1}{F_2^2})dF_2dF_2 \\ &= \frac{1}{F_2}\sigma_2F_2dW_2 + \frac{1}{2}(-\frac{1}{F_2^2}(\sigma_2F_2dW_2)^2 \\ &= \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2dt : \end{aligned}$$

$$d\log F_2(t) = \sigma_2(t)dW_2(t) - \frac{1}{2}\sigma_2(t)^2dt.$$

Integrate both sides:

$$\log F_2(T) - \log F_2(0) = \int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2dt :$$

$$F_2(T) = F_2(0)\exp\{\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2dt\}. \qquad \textbf{[8]}$$

The distribution of the random variable in the exponent is *Gaussian*, since it is a stochastic integral of a deterministic function by a Brownian motion (recall: sums of independent Gaussians are Gaussian; limits of Gaussians are Gaussian). **[4]**

Compute its expectation: as the Itô integral has mean 0,

$$E[\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2dt] = -\frac{1}{2}\int_0^T \sigma_2(t)^2dt. \qquad \textbf{[4]}$$

The variance is

$$var(\int_0^T \sigma_2(t)dW_2(t) - \frac{1}{2}\int_0^T \sigma_2(t)^2dt) = var(\int_0^T \sigma_2(t)dW_2(t))$$

(as the second term is deterministic)

$$= E[(\int_0^T \sigma_2(t)dW_2(t))^2] \qquad \text{(as the mean is 0)}$$

$$= \int_0^T \sigma_2(t)^2 dt, \qquad \text{(by Itô's isometry: MATL480, V.5).} \qquad [\mathbf{5}]$$

Summarising,

$$I(T) := \int_0^T \sigma_2(t) dW_2(t) - \frac{1}{2} \int_0^T \sigma_2(t)^2 dt \sim m + VN(0,1)$$

(here '$\sim m + VN(0,1)$' is shorthand for 'is distributed as $m + V$ times a $N(0,1)$ – a standard normal random variable'), where

$$m = -\frac{1}{2} \int_0^T \sigma_2(t)^2 dt, \qquad V^2 = \int_0^T \sigma_2(t)^2 dt.$$

That is,

$$F_2(T) = F_2(0) \exp\{I(T)\} = F_2(0) e^{m+VZ}, \qquad Z \sim N(0,1). \qquad [\mathbf{4}]$$

[Seen – lectures]

Q5. *Ornstein-Uhlenbeck (OU)/Vasicek (Vas) process.*
(i) The OU SDE $dV = -\kappa V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\kappa V dt$ term is *frictional drag*, and κ is the *inverse relaxation time*; the σdW term is *noise*, and σ is the *volatility*. **[2]**
(ii) $e^{-\kappa t}$ solves the corresponding homogeneous DE $dV = -\kappa V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\kappa t}$. Then

$$dV = -\kappa Ce^{-\kappa t}dt + e^{-\kappa t}dC = -\kappa V dt + e^{-\kappa t}dC,$$

so V is a solution of (OU) if $e^{-\kappa t}dC = \sigma dW$, $dC = \sigma e^{\kappa t}dW$, $C = c + \sigma\int_0^t e^{\kappa u}dW$. So with initial velocity v_0, $V = e^{-\kappa t}C$ is

$$V = v_0 e^{-\kappa t} + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}dW_u. \qquad [4]$$

(iii) V comes from W, Gaussian, by linear operations, so is Gaussian.
V_t has mean $v_0 e^{-\kappa t}$, as $E[e^{\kappa u}dW_u] = \int_0^t e^{\kappa u}E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\kappa t}\int_0^t e^{\kappa u}dW_u)^2] = \sigma^2 e^{-2\kappa t}\int_0^t (e^{\kappa u})^2 du$$

$$= \sigma^2 e^{-2\kappa t}[e^{2\kappa t} - 1]/(2\kappa) = \sigma^2[1 - e^{-2\kappa t}]/(2\kappa).$$

So V_t has distribution $N(v_0 e^{-\kappa t}, \sigma^2(1 - e^{-2\kappa t})/(2\kappa))$. **[4]**
(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which is

$$\sigma^2 E[e^{-\kappa t}\int_0^t e^{\kappa v}dW_v . e^{-\kappa(t+u)}(\int_0^t + \int_t^{t+u})e^{\kappa w}dW_w].$$

By independence of Brownian increments, $\int_t^{t+u}$ contributes 0, so by above

$$cov(V_t, V_{t+u}) = e^{-\kappa u}var(V_t) = \sigma^2 e^{-\kappa u}[1 - e^{-2\kappa t}]/(2\kappa) \to \sigma^2 e^{-\kappa u}/(2\kappa) \quad (t \to \infty).$$

[4]
(v) V is Markov (a diffusion), being the solution of the SDE (OU). The limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\kappa))$ (the *Maxwell-Boltzmann distribution* of Statistical Mechanics). As only the time-difference u survives the passage to the limit $t \to \infty$, the limit process is stationary; it is also Gaussian, and Markov, by above. **[3]**
(vi) The process shows *mean reversion* – a strong push towards the central

value. This is characteristic of interest rates (under normal conditions – post-Crash, interest rates have been stuck at just above zero – unprecedented). The financial relevance is to the *Vasicek model* of interest-rate theory. **[4]**

(vii) The Vasicek model is widely used because it is analytically tractable, and easy to interpret. Its main drawbacks both stem from its Gaussianity (as do its main advantages!):

(a) negative interest rates;

(b) poor fit to market data: tails too thin, symmetric rather than skew, etc.

In addition:

(c) One-factor models are not capable of capturing all relevant aspects; one needs at least a two- (or three-) factor model, and the Vasicek model does indeed extend easily to higher factors. **[4]**

[Seen, lectures.]

Q6. *Dupire's formula*

Theorem (Dupire's formula). (i) Writing the density of the forward rate F_t at time t as $\phi(t,x)$, the call price $C(T,K)$ satisfies

$$\partial C(T,K)/\partial T = \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K).$$

(ii) The *local volatility* $\sigma(T,K)$ is completely specified by the call-price $C(.,.)$ (via its derivatives) by Dupire's formula,

$$\sigma(T,K) = \frac{1}{K}\sqrt{\frac{2\partial C(T,K)/\partial T}{\partial^2 C(T,K)/\partial K^2}}.$$

Proof. Suppose we have an option on the forward rate $F(T)$ (or F_T for short), with payoff function h and expiry T. For $t \in [0,T]$, if

$$v(t,x) := E[h(F_T)|F_t = x],$$

$$\begin{aligned} E[h(F_T)] &= E[E[h(F_T)|F_t = x]] \qquad \text{(tower property)} \\ &= \int_0^\infty v(t,x)\phi(t,x)dx, \end{aligned}$$

as F_t has density $\phi(t,x)$. Now the LHS is *independent* of t. Hence, so too is the RHS: differentiating under the integral sign w.r.t. t as above,

$$0 = \int \frac{\partial v}{\partial t}\phi dx + \int v\frac{\partial \phi}{\partial t}dx. \qquad [\mathbf{4}]$$

Now, v satisfies the*Kolmogorov backward equation* (*Fokker-Planck equation*):

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma(t,x)^2 x^2 \frac{\partial^2 v}{\partial x^2} = 0, \qquad v(T,x) = h(x) \qquad (FoPl)$$

(given). By $(FoPl)$, we can substitute for the $\partial v/\partial t$ term in the above, to obtain (writing v' for $\partial v/\partial x$, etc.)

$$0 = -\frac{1}{2}\int(\sigma^2 x^2 \phi)v'' dx + \int v\frac{\partial \phi}{\partial t}dx. \qquad (*) \quad [\mathbf{4}]$$

Integrate the first integral by parts: the integrated term vanishes (at 0 because of the x^2, at infinity because the other factors decay fast enough):

$$\int(\sigma^2 x^2\phi)v''dx = \int(\sigma^2 x^2\phi)dv' = -\int(\sigma^2 x^2\phi)'v'dx = -\int(\sigma^2 x^2\phi)'dv.$$

Integrate by parts again: again the integrated terms vanish, giving

$$\int (\sigma^2 x^2 \phi) dv = \int v(\sigma^2 x^2 \phi)'' dx.$$

Substituting this in $(*)$,

$$0 = \int (\frac{1}{2}(\sigma^2 x^2 \phi - \frac{\partial \phi}{\partial \tau}) v dx.$$

But the payoff h, and so the conditional density v, is arbitrary. So the integrand here must vanish, giving the *forward equation*

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma(t,x)^2 x^2 \phi). \qquad (ForEq) \quad [\mathbf{7}]$$

Suppose now that the option above is a call C with strike K. Then

$$C(T,K) = E[(F-K)_+] = E[(F-K)I(F>K)] = \int_K^\infty (x-K)\phi(t,x)dx.$$

So, first differentiating under the integral sign w.r.t. K,

$$\partial C(T,K)/\partial K = -\int_K^\infty \phi(T,x)dx$$

(the $(x-K)$ term vanishes at the lower limit). So

$$\partial^2 C(T,K)/\partial K^2 = \phi(T,K). \qquad (**) \quad [\mathbf{4}]$$

Next, differentiate w.r.t. T under the integral sign and use $(ForEq)$:

$$\begin{aligned} \frac{\partial C(T,K)}{\partial T} &= \int_K^\infty (x-K) \frac{\partial \phi(T,x)}{\partial T} dx \\ &= \int_K^\infty (x-K).\frac{1}{2}(\sigma^2 x^2 \phi)'' dx \qquad \text{(by } (ForEq)) \\ &= -\frac{1}{2}\int_K^\infty (\sigma^2 x^2 \phi)' dx = -\frac{1}{2}\int_K^\infty d(\sigma^2 x^2 \phi) \qquad \text{(integrating by parts)} \\ &= \frac{1}{2}\sigma(T,K)^2 K^2 \phi(T,K) \qquad \text{(lower limit, hence the -),} \end{aligned}$$

performing the integration. This gives (i). **[3]**

Then (ii) follows from (i) and $(**)$. **[3]**

[Seen – lectures] NHB