ullintsoln4 pm   6.3.2019

SOLUTIONS 4 (Week 6). 6.3.29019

Q1. *Rho.*
(i) *Rho for calls.*

With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$, $\tau := T - t$ the time to expiry, the Black-Scholes call price is, with d_1, d_2 as given,

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2). \qquad (BS)$$

So as $d_2 = d_1 - \sigma\sqrt{\tau}$,

$$\phi(d_2) = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad K e^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*)$$

(ii) Differentiating (BS) partially w.r.t. r gives, by $(*)$,

$$\begin{aligned} \rho := \partial C/\partial r &= S\phi(d_1)\partial d_1/\partial r - K e^{-r\tau}\phi(d_2)\partial d_2/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(d_1 - d_2)/\partial r + K\tau e^{-r\tau}\Phi(d_2) \\ &= S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial r + K\tau e^{-r\tau}\Phi(d_2) = K\tau e^{-r\tau}\Phi(d_2) : \end{aligned}$$

$$\rho > 0.$$

(iii) *Financial interpretation.*

As r increases, cash becomes more attractive compared to stock. So stock *buyers* have a ‘buyer’s market’, favouring them. So for *calls* (options to buy), $\rho > 0$.
(iv) *Rho for puts.*

By put-call parity, $S + P - C = K e^{-r\tau}$:

$$\partial P/\partial r = \partial C/\partial r - K\tau e^{-r\tau} = -K\tau e^{-r\tau}[1 - \Phi(d_2)] = -K\tau e^{-r\tau}\Phi(-d_2) < 0.$$

(v) *Financial interpretation.*

As above: as r increases, stock *sellers* also operate in a buyer's market, but this is against them. So for *puts* (options to sell), $\rho < 0$.

(vi) *American options.*

All this extends to American options,via the *Snell envelope*, which is *order-preserving*. The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As r increases, the Z-terms increase for calls (rho is positive for European calls). As the Zs increase, the Us increase (above: backward induction on n – dynamic programming, as usual for American options). Combining: as r increases, the U-terms increase. So rho is also positive for American calls. Similarly, rho is negative for American puts. [Similar to 'vega positive', done in Problems] NHB