ullintsoln5

SOLUTIONS 5, Week 6, 6.3.2019

(*Theta*). Given
$$Ke^{-r(T-t)}\phi(d_2) = S\phi(d_1) : \qquad (*)$$
(i) *Calls*. Given the Black-Scholes formula for the price c_t of European calls,
$$c_t = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$
$$d_{1,2} := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t} : \qquad d_2 = d_1 - \sigma\sqrt{T-t} :$$
(a) Differentiating and using $(*)$: as
$$\partial(d_1 - d_2)/\partial t = \partial(\sigma\sqrt{T-t})/\partial t = -\frac{1}{2}\sigma/\sqrt{T-t} :$$
$$\Theta = \partial c_t/\partial t = S\phi(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\phi(d_2)\frac{\partial d_2}{\partial t} :$$
$$\Theta = Ke^{-r(T-t)}[\phi(d_2)\frac{\partial(d_1 - d_2)}{\partial t} - r\Phi(d_2)] = -Ke^{-r(T-t)}[\phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}} + r\Phi(d_2)] :$$
$$\Theta < 0.$$
(b) Interpretation: an option is (partly) an insurance against future uncertainty. As time passes, there is less future (till expiry) to protect against, so such protection becomes less valuable. (ii) *Puts*. Given the corresponding BS formula for European puts,
$$p_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1),$$
(a) As above, as $\phi(-x) = \phi(x)$,
$$\Theta = \partial p_t/\partial t = rKe^{-r(T-t)}\Phi(-d_2) + Ke^{-r(T-t)}\phi(d_2)\frac{\partial(-d_2)}{\partial t} - S\phi(d_1)\frac{\partial(-d_1)}{\partial t} :$$
$$\Theta = Ke^{-r(T-t)}[r\Phi(-d_2) + \phi(d_2)\frac{\partial(d_1 - d_2)}{\partial t}] = Ke^{-r(T-t)}[r\Phi(-d_2) - \phi(d_2)\cdot\frac{\frac{1}{2}\sigma}{\sqrt{T-t}}].$$
This can change sign! (b) The situation with puts is different, because of the different role of the strike K (fixed, while S varies). But for large enough K (when a put option – the right to *sell* at price K – will be deeply in the money), the option stands to make a large profit – so the nearer this is to being realised, the better. NHB