ull1b.tex **Week 1: pm, 27.9.2017**

Options (continued).

Because the value of an option at expiry is so sensitive to price – it reflects movements in the price of the underlying in exaggerated form – the holding (or more generally, trading) of options and other derivatives presents greater opportunities for profit (and indeed, for loss) than trade in the underlying (this is why speculators buy options!). They are correspondingly more risky than the underlying.

One of the main insights of the fundamental work of Black and Scholes was that one can (at least in the most basic model) *hedge* against meeting a contingent claim by *replicating* it: constructing a portfolio, adjusted or rebalanced as time unfolds and new price information comes in, whose pay-off *is* the amount of the contingent claim.

6. Arbitrage.

Economic agents go to the market for various reasons. One the one hand, companies may wish to insure, or *hedge*, against adverse price movements that might affect their core business. On the other hand, *speculators* may be uninterested in the specific economic background, but only interested in making a profit from some financial transaction. The relation between hedging ('good') and speculation ('bad') is to some extent symbiotic (one cannot lay off a risk unless someone else is prepared to take it on, and why should he unless he expects to make money by doing so). Nevertheless, one feels that it should not be possible to extract money from the market without genuinely engaging in it, by taking *risk*: all business activity is risky. Indeed, were it possible to do so, people would do so – in unlimited quantities, thus sucking money parasitically out of the market, using it as a 'money-pump'. This would undermine the stability and viability of the market in the long run – and in particular, make it impossible for the market to be in *equilibrium*.

The usual theoretical view of modelling markets as NA is not so much that arbitrage opportunities do not exist, but rather that if they do exist in any sizeable quantity, people will rush to exploit them, and by doing so will dissipate them – 'arbitrage them away'.

Used in this broad sense, the term covers financial activity of many kinds, including trade in options, futures and foreign exchange. However, the term arbitrage is nowadays also used in a narrower and more technical sense. Financial markets involve both riskless (bank account) and risky (stocks, etc.) assets. To the investor, the only point of exposing oneself to risk is the opportunity, or possibility, of realising a greater profit than the riskless procedure of putting all one's money in the bank (the mathematics of which – compound interest – does not require a degree or MSc course!). Generally speaking, the greater the risk, the greater the return required to make investment an attractive enough prospect to attract funds. Thus, for instance, a clearing bank lends to companies at higher rates than it pays to its account holders. The companies' trading activities involve risk; the bank tries to spread the risk over a range of different loans, and makes its money on the difference between high/risky and low/riskless interest rates.

It is usually better to work, not in face-value or nominal terms, but in *discounted* terms, allowing for the exponential growth-rate e^{rt} of risklessly invested money. So, profit and loss are generally reckoned against this discounted benchmark.

The above makes it clear that a market with arbitrage opportunities would be a disorderly market – too disorderly to model. The remarkable thing is the converse. It turns out that the minimal requirement of absence of arbitrage opportunities is enough to allow one to build a model of a financial market which – while admittedly idealised (frictionless market – no transaction costs, etc.) – is realistic enough both to provide real insight and to handle the mathematics necessary to price standard options (Black-Scholes theory). We shall see that arbitrage arguments suffice to determine prices – the *arbitrage pricing technique* (APT).

Short-selling.

First, consider a riskless asset (bank account), with interest-rate $r > 0$. If our bank deposit is positive, we *lend* money and *earn* interest at rate r. If our bank deposit is negative (overdraft), we *borrow* money and *pay* interest. [We assume for simplicity that we pay interest also at rate r, though in practice of course it will be at some higher rate $r' > r$. Models taking these different interest-rates into account are topical at research level; we omit them here – see VI.5].

In many markets, risky assets such as stocks may be treated in the same way. We may have a positive or zero holding – or a *negative holding* (notionally borrowing stock, which we will be obliged to repay – or repay its current value). In particular, we may be allowed to *sell stock we do not own.* This is called *short-selling*, and is perfectly legal (subject to appropriate regulation) in many markets. Think of short-selling as borrowing. Not only is short-selling both routine and necessary in some contexts, such as foreign exchange and commodities futures, it simplifies the mathematics. So

we assume, unless otherwise specified, no restriction on short-selling. By extension, we call a portfolio, or position, *short* in an asset if the holding of the asset is negative, *long* if the holding of the asset is positive.
Note. It turns out that in some important contexts – such as the Black-Scholes theory of European and American calls – short-selling can be avoided. In such cases, it is natural and sensible to do so: see Ch. VI.

7. Put-Call Parity.

Just as long and short positions are diametrical opposites, so are call and put options. We now use arbitrage to show how they are linked.

Suppose there is a risky asset, value S (or S_t at time t), with European call and put options on it, value C, P (or C_t, P_t), with expiry time T and strike-price K. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for the value of this portfolio. So

$$\Pi = S + P - C \qquad (S\text{: long asset; } P\text{: long put; } -C\text{: short call}).$$

Recall that the payoffs at expiry are:

$$\begin{cases} \text{call } C\text{:} & \max(S-K,0), \quad \text{or} \quad (S-K)_+, \\ \text{put } P\text{:} & \max(K-S,0), \quad \text{or} \quad (K-S)_+. \end{cases}$$

So the value of the above portfolio at expiry is

$$\begin{cases} S + 0 - (S-K) = K & \text{if } S \geq K, \\ S + (K-S) - 0 = K & \text{if } K \geq S, \end{cases}$$

namely K (see Problems 1b Q1 for the quicker way to do this). This portfolio thus guarantees a payoff K at time T. How much is it worth at time t?

Short answer (correct, and complete): $Ke^{-r(T-t)}$, because it is financially equivalent to cash K, so has the same time-t value as cash K.

Longer answer (included as an example of arbitrage arguments). The riskless way to guarantee a payoff K at time T is to deposit $Ke^{-r(T-t)}$ in the bank at time t and do nothing. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time T my portfolio yields K (above), while my bank debt has grown to K. I clear my cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale

at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio (short)* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, my bank deposit has grown to K, and I again clear my cash account – use this to meet my obligation K on the portfolio I sold short, again locking in my earlier riskless profit. So the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make a riskless profit) – which they will take! Thus
(i) The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

This link between the prices of the underlying asset S and call and put options on it is called *put-call parity*.
(ii) The value of the portfolio $S+P-C$ is the *discounted value of the riskless equivalent*. This is a first glimpse at the central principle, or insight, of the entire subject of option pricing. But in general, we will have 'risk-neutral' in place of 'riskless'; see I.8 below, Ch. IV and Ch. VI.
(iii) Arbitrage arguments, although apparently qualitative, have quantitative conclusions, and allow one to calculate precisely the rational price – or *arbitrage price* – of a portfolio. The put-call parity argument above is the simplest example – though a typical one – of the *arbitrage pricing technique*.
(iv) The arbitrage pricing technique (APT) is due to S. A. Ross in 1976-78 (details in [BK], Preface). Put-call parity has a long history (see Wikipedia).
Note. 1. History shows both that arbitrage opportunities exist (or are sought) in the real world and that the exploiting of them is a delicate matter. The collapse of Barings Bank in 1995 (the UK's oldest bank, and bankers to HMQ) was triggered by unauthorised dealings by one individual, who tried and failed to exploit a fine margin between the Singapore and Osaka Stock Exchanges. The leadership of Barings at that time thought that he had discovered a clever way to exploit price movements in either direction. This is obviously impossible on theoretical grounds, to anyone who knows any Physics. See Problems 1b Q2 (key phrases: perpetual motion machine; Maxwell's demon; Second Law of Thermodynamics; entropy).
2. Major finance houses have an *arbitrage desk*, where their *arbs* work.

8. An Example: Single-Period Binary Model.

We consider the following simple example, taken from
[CRR] COX, J. C., ROSS, S. A. & RUBINSTEIN, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263.
We shall see later that (in a sense) this captures Black-Scholes theory in microcosm.

For definiteness, we use the language of foreign exchange. Our risky asset will be the current price in Swiss francs (SFR) of (say) 100 US \$, supposed $X_0 = 150$ at time 0. Consider a call option with strike price $K = 150$ at time T. The simplest case is the binary model, with two outcomes: suppose the price X_T of 100 \$ at time T is (in SFR)

$$X_T = \begin{cases} 180 & \text{with probability } p, \\ 90 & \text{with probability } 1-p. \end{cases}$$

The payoff H of the option will be $30 = 180 - 150$ with probability p, 0 with probability $1-p$, so has expectation $EH = 30p$. This would seem to be the fair price for the option at $t = 0$, or allowing for an interest-rate r and discounting, we get the value

$$V_0 = E(\frac{H}{1+r}) = \frac{30p}{1+r}.$$

Take for simplicity $p = \frac{1}{2}$ and $r = 0$ (no interest): the naive, or expectation, value of the option at time 0 is

$$V_0 = 15.$$

The *Black-Scholes value* of the option, however, is different. To derive it, we follow the Black-Scholes prescription (Ch. IV, VI):
(i) First replace p by p^* so that the price, properly discounted, *behaves like a fair game*:

$$X_0 = E^*(\frac{X_T}{1+r}).$$

That is,

$$150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90);$$

for $r = 0$ this gives $60 = 90p^*$ or $p^* = 2/3$.
(ii) Now compute the fair price of the expected value in this new model:

$$V_0 = E^*(\frac{H}{1+r}) = \frac{30p^*}{1+r};$$

for $r = 0$ this gives the Black-Scholes value as $V_0 = 20$.
Justification: it works! – as the arbitrage constructed below shows. For simplicity, take $r = 0$.
We *sell* the option at time 0, for a price $\pi(H)$, say. We then prepare for the resulting contingent claim on us at time T by the option holder by using the following strategy:

Sell the option for $\pi(H)$	$+\pi(H)$
Buy \$33.33 at the present exchange rate of 1.50	-50
Borrow SFR 30	$+30$
Balance	$\pi(H) - 20$.

So our balance at time 0 is $\pi(H) - 20$. At time T, two cases are possible:
(i) The dollar has risen:

Option is exercised (against us)	-30
Sell dollars at 1.80	$+60$
Repay loan	-30
Balance	0.

(ii) The dollar has fallen:

Option is worthless	0.00
Sell dollars at 0.90	$+30$
Repay loan	-30
Balance	0.

So the balance at time T is zero in both cases. The balance $\pi(H) - 20$ at time 0 should thus also be zero, giving the Black-Scholes price $\pi(H) = 20$ as above. For, *any other price* gives an arbitrage opportunity. Argue as in put-call parity in §4: if the option is offered too cheaply, buy it; if it is offered too dearly, write it (the equivalent for options to 'sell it short' for stock). Thus any other price would offer an arbitrageur the opportunity to extract a riskless profit, by appropriately buying and selling (Swiss francs, US dollars and options) so as to exploit your mis-pricing.

The same argument with interest-rate r also applies: divide everything through by $1 + r$.
Note. This argument, and result, are **independent** of p, the 'real' probability, and depend instead **only** on this 'fictitious' new probability, p^* (which is called the *risk-neutral* or *risk-adjusted* probability.

The example above is highly instructive. First, it clearly represents the simplest possible non-trivial case: only two time-points (with one time-period between them, hence the 'single-period' of the title), and only two possible outcomes (hence the 'binary' of the title). Secondly, it shows that there is a theory hidden here, which gives us a definite prescription to follow (and some surprises, such as not involving the 'real' probability p above). This prescription is simple to implement, and can be justified by explicitly constructing an arbitrage to exploit doing anything else [if the option is offered for sale too cheaply, buy it, if too dearly, write it]. This theory is the Black-Scholes theory, which we consider in detail in Chapters IV and VI. The technical key to the Black-Scholes prescription is the introduction of p^* and its associated expectation operator E^*. In technical language, this is the *equivalent martingale measure*. Now each of these three terms needs full introduction. We shall talk about measures in II.1 below, about equivalent measures in II.4, and martingales in III.3 and V.2. We stress: the Black-Scholes theory – that is, rational option pricing – cannot be done without all these concepts. This is why we need Chapter II on the necessary background on measure theory, and Chapters III and V on the necessary background on stochastic processes.

There are basically three options open to those teaching, and learning, how to price options etc.

1. One can avoid measure theory altogether (cf. [CR]). This is technically possible rigorously in the discrete-time setting of Ch. III – though at greater length, because the key concepts cannot be addressed explicitly. It is also possible non-rigorously in continuous time, using partial differential equations (PDEs) rather than martingales.
2. One can learn measure theory first – say, from the excellent book [W]. This, however, puts the subject beyond the reach of most people who need it and use it in practice – and beyond reach of most of this audience.
3. One can do as we shall do (and as [BK] does): state what we need from measure theory, and use its language, concepts, viewpoint and results, without proving anything. This makes good sense: the constructions and proofs of measure theory are quite hard (say, final year undergraduate or first-year postgraduate level for good mathematics students with a bent for analysis – quite a select group!). Using measure theory taking its results for granted, however, is quite easy, as we shall see.

9. Complements

1. Types of risk.

Institutions encounter risks of various types. These include:

Market risk.

This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer).

Credit risk.

This is the risk that counter-parties to one's financial transactions may default on their obligations.

When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. This raises issues of *moral hazard*, below.

Operational risk.

This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions (the failure of the Taurus clearing system on the London Stock Exchange was one example); fraudulent or unauthorised trading made possible by inadequate supervision; etc.

Liquidity risk.

This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up.

Model risk.

To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful.[1] Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models.

2. Risk management. The problems of 2007/8 on have made the importance

[1]George E. Box, 1919-2013, British statistician

of risk management obvious. For an excellent book-length treatment, see e.g. [MFE] A. J. McNEIL, R. FREY & P. EMBRECHTS: *Quantitative risk management: Concepts, techniques, tools*. Princeton UP, 2005.

We know from Markowitz that we should have a balanced portfolio, with lots of negative correlation. The danger is *large* losses, quantified by the *tails* of the joint distribution of our portfolio. We diversify so that what we lose on the swings we gain on the roundabouts. Two comments:
(a) Whether this works for large losses depends on the tail properties of the joint distribution. It does *not* work if this is normal – as it is in the benchmark Black-Scholes model.
(b) When the whole market is falling – as in a financial crisis – none of the risk-management techniques useful under normal market conditions work.
3. Moral hazard. Before the limited liability company, if one defaulted, one was liable to the *whole* of the loss incurred by one's counter-party. This made trading very dangerous (the early traders were called merchant adventurers) – all the more as insurance had not developed by then.[2]

Limited liability was what made ordinary people willing to undertake the risks of trading, and so paved the way for the development of modern business, commerce, capitalism etc.

The moral hazard here is the possibility of *gambling with other people's money* (see Kay's book [K2], Week 0). If it works, fine. If not, walk away (writing off one's limited liability) and leave them to bear the loss.

Bankruptcy law varies from country to country, and is too complicated to pursue here. But one sees moral hazard where it concerns us in, e.g.:
(a) start-ups of hedge funds (or, dot-com companies);
(b) aggressive traders – who (for the sake of their bonuses) gamble with their careers – but with other people's money;
(c) credit rating – where the credit rating agencies had a financial incentive to pass as AAA some highly questionable financial asset, etc.
4. Securitization. This term covers the drive in recent years to seek out new financial markets by identifying risks that people might want to cover themselves against, and creating new financial derivatives that can be sold to address this perceived need. These derivatives too could be traded, etc. The upshot was an explosion of trade in increasingly artificial financial products,

[2]Lloyds of London predates limited liability. The Lloyds participants – "names" – had unlimited liability. Many were driven into personal bankruptcy in the Lloyds scandals of the 90s. See Google for the ghastly details.

developed by the R&D departments of the financial institutions. By 2007/8 the leaders of these institutions did not understand these products – could not price them, and could not value their holdings of them (above).

One specific trigger of the US crash in 2007 was the explosive growth in sub-prime mortgages. These were granted to people who would not have qualified as financially sound enough to get a mortgage previously, but who wanted to buy their own house. This new and profitable market proved irresistible to US banks – leading to a great house-price bubble, which burst (as bubbles do) in 2007. The knock-on effects hit the UK in 2008 (Northern Rock, etc.). The real damage of this failure of the financial sector has been its devastating and ongoing consequences on the real economy.

5. Macro-prudential issues. As the above illustrates, financial matters are too important to be left to financiers. Proper regulation is vital.

6. Forwards and futures. Forwards are agreements between buyer and seller made now, but concerning delivery in the future. They are not traded. Futures are options on things that will come to market in the future (next year's grain crop, for example), and these are traded (extensively). There are good accounts in Hull's books, [H1], [H2].

7. OTC and exchange-traded contracts. OTC – "over-the-counter" – denotes a transaction made between an individual buyer and an individual seller. As options on standard transactions develop, these are assets themselves that can be traded in exchanges (e.g., the CBOE, which opened in 1973: I.3).

8. Marking to market. This is a system whereby the exchanges cover themselves and their clients against the risk of large losses. If one party to a trade is, on current market prices, exposed to a potentially heavy loss, a *margin call* will be required by the exchange. It is margin calls that actually trigger many financial failures (but limit the losses of the counter-parties)[3].

9. Forex (FX). Forex is an abbreviation for foreign exchange. International trade involves more than one currency' currencies move against each other. There is a vast market in derivatives to cover the risks involved.

10. Swaps. From Hull [H2] Ch. 5: "Swaps are private agreements between two companies to exchange cash flows in the future ... The first swap contracts were negotiated in 1981. Since then the market has grown very rapidly. ..." There are even options on swaps – *swaptions* – etc.

[3]The 2011 film Margin Call, starring Kevin Spacey, is about a finance house about to collapse in this way.

Prelude to Ch. II: Integration and area (cf. PfS Lecture 1, SP L1)

We shall mainly deal with area, as this is two-dimensional. We can draw pictures in two dimensions, and our senses respond to this; paper, whiteboards and computer screens are two-dimensional. By contrast, one-dimensional pictures are much less vivid, while three-dimensional ones are harder (they need the mathematics of perspective) – and dimensions higher than four are harder still.

Area.

1. *Rectangles*, base b, height h: area $A := bh$.
2. *Triangles.* $A = \frac{1}{2}bh$.

Proof: Drop a perpendicular from vertex to base; then extend each of the two triangles formed to a rectangle and use 1. above.

3. *Polygons.* Triangulate: choose a point in the interior; connect it to the vertices. This reduces A to the sum of areas of triangles; use 2. above.
4. *Circles.* We have a choice:

(a) Without calculus. Decompose the circle into a large number of equi-angular sectors. Each is approximately a triangle; use 2. above [the approximation boils down to $\sin\theta \sim \theta$ for θ small].

(b) With calculus and plane polar coordinates. Use $dA = dr.rd\theta = rdrd\theta$: $A = \int_0^r \int_0^{2\pi} rdrd\theta = \int_0^r rdr. \int_0^{2\pi} d\theta = \frac{1}{2}r^2.2\pi = \pi r^2$.

Note. The ancient Greeks essentially knew integral calculus – they could do this, and harder similar calculations [volume of a sphere $V = \frac{4}{3}\pi r^3$; surface area of a sphere $S = 4\pi r^2 dr$, etc.; note $dV = Sdr$].

What the ancient Greeks did not have is *differential calculus* [which we all learned first!] Had they had this, they would have had the idea of velocity, and differentiating again, acceleration. With this, they might well have got Newton's Law of Motion, Force = mass $\times$ acceleration. This triggered the Scientific Revolution. Had this happened in antiquity, the world would have been spared the Dark Ages and world history would have been completely different!

5. *Ellipses*, semi-axes a, b. Area $A = \pi ab$ (w.l.o.g., $a > b$).

Proof: cartesian coordinates: $dA = dx.dy$.

Reduce to the circle case: compress ['squash'] the x-axis in the ratio b/a [so $dx \mapsto dx.b/a$, $dA \mapsto dA.b/a$]. Now the area is $A = \pi b^2$, by 4. above. Now 'un-squash': dilate the x-axis in the ration a/b. So $A \mapsto A.a/b = \pi b^2.a/b = \pi ab$.

Fine – what next? We have already used *both* the coordinate systems to hand. There is no general way to continue this list.

The only general procedure is to superimpose finer and finer sheets of graph paper on our region, and count squares ('interior squares' and 'edge squares'). This yields numerical approximations – which is all we can hope for, and all we need, in general.

The question is whether this procedure always works. Where it is clearly most likely to fail is with highly irregular regions: 'all edge and no middle'.

It turns out that this procedure does *not* always work; it works for *some but not all* sets – those whose structure is 'nice enough'[4]. This goes back to the 1902 thesis of Henri LEBESGUE (1875-1941):
H. Lebesgue: Intégrale, longueur, aire. *Annali di Mat.* **7** (1902), 231-259.
Similarly in other dimensions. So: some but not all sets have a length/area/volume. Those which do are called *(Lebesgue) measurable*; length/area/volume is called *(Lebesgue) measure*; this subject is called Measure Theory.

We first meet integration in just this context – finding areas under curves (say). The 'Sixth Form integral' proceeds by dividing up the range of integration on the x-axis into a large number of small subintervals, $[x, x + dx]$ say. This divides the required area up into a large number of thin strips, each of which is approximately rectangular; we sum the areas of these rectangles to approximate the area.

This informal procedure can be formalised, as the *Riemann integral* (G. F. B. RIEMANN (1826-66) in 1854). This (basically, the Sixth From integral formalised in the language of epsilons and deltas) is part of the undergraduate Mathematics curriculum.

We see here the essence of calculus (the most powerful single weapon in mathematics, and indeed in science). If something is reasonably smooth, and we break it up finely enough, curves look straight, so we can handle them. We make an error by this approximation, but when calculus applies, this error can be made arbitrarily small, so the approximation is effectively exact.
Example: We do this sort of thing automatically. If in a discussion of global warming we hear an estimate of polar ice lost, this will translate into an estimate of increase in sea level (neglecting the earth's curvature).
Note. The 'squashing' argument above was deliberately presented informally. It can be made quite precise – but this needs the mathematics of *Haar measure*, a fusion of Measure Theory and Topological Groups.

[4]The existence of *non-measurable sets* is bound up with the axioms of Set Theory. We assume the Axiom of Choice (AC) – E. Zermelo, 1904.