

### SOLUTIONS 3b. 18.10.2017

Q1 *Doubling strategy.* (i) With  $N$  the number of losses before the first win:

$$P(N = k) = P(L, L, \dots, L(k \text{ times}), W) = \left(\frac{1}{2}\right)^k \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{k+1}.$$

That is,  $N$  is geometrically distributed with parameter  $1/2$ . As

$$\sum_{k=0}^{\infty} P(N = k) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2} / \left(1 - \frac{1}{2}\right) = 1,$$

$P(N < \infty) = 1$ :  $N < \infty$  a.s. So one is certain to win eventually.

(ii) Let  $S_n$  be one's fortune at time  $n$ . When  $N = k$ , one has losses at trials  $1, 2, 3, \dots, k$ , with losses  $1, 2, 4, \dots, 2^{k-1}$ , followed by a win at trial  $k + 1$  (of  $2^k$ ). So one's fortune then is

$$2^k - (1 + 2 + 2^2 + \dots + 2^{k-1}) = 2^k - (2^k - 1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is  $+1$  (which, by (i), one is certain to win eventually).

(iii)  $N$  has PGF

$$\begin{aligned} P(s) &:= E[s^N] = \sum_{n=0}^{\infty} s^n P(N = n) = \sum_{n=0}^{\infty} s^n \cdot \left(\frac{1}{2}\right)^{n+1} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}s\right)^n = \frac{1}{2} / \left(1 - \frac{1}{2}s\right) = 1/(2 - s) : \end{aligned}$$

$$P'(s) = E[Ns^{N-1}] = (2 - s)^{-2}; \quad P'(1) = E[N] = 1.$$

So the mean time the game lasts is 1.

(iv) As with the simple random walk (Q2 below): this is an impossible strategy to use in reality, for two reasons:

(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky.

(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice.

Q2 *First-passage time for simple random walk (SRW).*

Let  $F(s) := s^T = \sum_1^\infty P(T = n)s^n = \sum_1^\infty f_n s^n$  be the PGF of  $T$  ( $= T_1$ , the first passage time to 1). Since the first-passage time  $T_2$  to 2 is the sum of the first-passage times from 0 to 1 (PGF  $F$ ) and from 1 to 2 (PGF  $F$  again), and these are independent (they involve disjoint blocks of independent tosses),  $T_2$  has PGF  $F_2(s) := E[s^{T_2}] = F(s)^2$ .

Condition on the outcome  $X_1$  of the first toss. If this is head (+1),  $T_1 = 1$ . If it is a tail (−1),  $T = 1 + U$ , where  $U$ , the first-passage time from −1 to 1, has PGF  $F_2(s) = F(s)^2$  as above. So

$$\begin{aligned} F(s) &:= E[s^T] = E[s^T | X_1 = +1]P(X_1 = +1) + E[s^T | X_1 = -1]P(X_1 = -1) \\ &= \frac{1}{2} \cdot s + \frac{1}{2} \cdot s F(s)^2 \end{aligned}$$

(as 1 has PGF  $s$ ). So  $F$  satisfies the quadratic

$$\frac{1}{2}sF(s)^2 - F(s) + \frac{1}{2}s = 0. \quad \text{So} \quad F(s) = \frac{1 \pm \sqrt{1-s^2}}{s}.$$

We need to take the  $-$  sign here (as  $F(s)$  contains no  $s^{-1}$  term):

$$F(s) = \frac{1 - \sqrt{1-s^2}}{s}.$$

(i) Put  $s = 1$ :  $F(1) = 1$ , so  $\sum_1^\infty P(T = n) = 1$ , so  $T < \infty$  a.s.

(ii)

$$F'(s) = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} - \frac{1}{s} \cdot \frac{\frac{1}{2}(-2s)}{\sqrt{1-s^2}} = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} + \frac{1}{\sqrt{1-s^2}}.$$

So  $F'(1) = E[T] = +\infty$ .

(iii) In particular,  $P(T = n) > 0$  for infinitely many  $n$  (indeed, for all odd  $n$ ). So no bound can be put on our maximum net loss before we realise our eventual gain.

This strategy is even more unrealistic than that in Q1: it has all the disadvantages there, plus another – infinite mean waiting time.

NHB