ullsoln4b.tex

SOLUTIONS 4b. 25.10.2017

Q1.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx = \int_{-\infty}^{\infty} e^{tx} . \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\} dx.$$

Make the substitution $u := (x-\mu)/\sigma$: $x = \mu + \sigma u$, $dx = \sigma du$:

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)} . \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\} du = e^{\mu t} . \int_{-\infty}^{\infty} e^{\sigma t u} . \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}u^2\} du.$$

Completing the square in the exponent on the right,

$$M(t) = e^{\mu t} . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[u^2 - 2\sigma t u]\} du$$

$$= e^{\mu t} . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}[(u-\sigma t)^2 - \sigma^2 t^2]\} du = e^{\mu t + \frac{1}{2}\sigma^2 t^2} . \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(u-\sigma t)^2\} du.$$

The integral on the right is 1 (a density integrates to 1 – of $N(\sigma t, 1)$ as it stands, or of $N(0,1)$ after the substitution $v := u - \sigma t$), giving

$$M(t) = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}.$$

Q2. (i) By Q1, $M_Y(t) = E[e^{tY}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$. Taking $t = 1$, $M_Y(1) = E[e^Y] = \exp\{\mu + \frac{1}{2}\sigma^2\}$. As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}.$$

(ii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations (with B Brownian motion)

$$dS = S(\mu dt + \sigma dB). \qquad (GBM)$$

This has solution (we quote this – from Itô's lemma – Ch. V)

$$S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normally distributed, so S_t is lognormal.

Q3. *The exponential martingale for Brownian motion.*

The MGF of $X \sim N(\mu, \sigma^2)$ is

$$E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}, \qquad (*)$$

by Q1. For $B = (B_t)$ Brownian motion (BM), write

$$M_t := \exp\{\theta B_t - \frac{1}{2}\theta^2 t\}.$$

Then with $\mathcal{F} = (\mathcal{F}_t)$ the Brownian filtration, for $s \leq t$,

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E[\exp\{\theta B_t - \frac{1}{2}\theta^2 t\}|\mathcal{F}_s] \\ &= E[\exp\{\theta(B_s + (B_t - B_s)) - \frac{1}{2}\theta^2 s - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s] \\ &= \exp\{\theta B_s - \frac{1}{2}\theta^2 s\}.E[\exp\{\theta(B_t - B_s)) - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s], \end{aligned}$$

taking out what is known. The first term on the right is M_s. The conditioning in the second term can be omitted, by independent increments of BM. But $B_t - B_s \sim N(0, t-s)$, which has MGF

$$E[\exp\{\theta(B_t - B_s)\}] = \exp\{\frac{1}{2}\theta^2(t-s)\}$$

(by $(*)$, with $\mu \mapsto 0$, $\theta^2 \mapsto t-s$, $t \mapsto \theta$). So the second term on RHS 1:

$$E[M_t|\mathcal{F}_s] = M_s.$$

So M is a martingale. //

NHB