MATL480 Mock EXAMINATION, 2016

Six questions, do four; 25 marks each

Q1. Discuss briefly:
(i) arbitrage;
(ii) completeness of markets;
(iii) equivalent martingale measures;
(iv) risk-neutral valuation.

Q2. Analyze the 'doubling strategy': when betting on tossing a fair coin, respond to losing by doubling the stakes.

Show that this leads to an eventual certain gain. Explain why this does not work in practice as a means of making money.

Q3. (i) Given a random variable Y with $E[|Y|] < \infty$ and a σ-field $\mathcal{C}$, define the conditional expectation $E[Y|\mathcal{C}]$ of Y with respect to $\mathcal{C}$.
(ii) What happens if $\mathcal{C}$ is the trivial σ-field? What happens if $\mathcal{C}$ is the whole σ-field?
(iii) Show that if $\mathcal{B}$, $\mathcal{C}$ are σ-fields with $\mathcal{B} \subset \mathcal{C}$, then applying both conditional expectations is the same as applying just one of them, either way round. Which one?
(iv) Show that the expectation of a conditional expectation is just the expectation.
(v) Explain what is meant by saying that a conditional expectation is a projection.

Q4. Given the Black-Scholes formula for a call price,

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \qquad (BS)$$

where

$$\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}, \qquad \Phi(x) := \int_{-\infty}^{x} \phi(u) du$$

are the standard normal density and distribution functions, $\tau := T - t$ the time to expiry,

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau},$$

show that
(i) $\phi(d_2) = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}$;
(ii) $e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}$;
(iii) $\phi(d_2) = \phi(d_1).(S/K).e^{r\tau}$: $\qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1)$.
Hence show that vega (the partial derivative of the option price with respect to the volatility σ) is positive. Obtain the same result for put option prices. Give the financial interpretation of these results.

Q5. (i) In the Black-Scholes model with riskless interest rate r and one risky stock with mean return rate μ and volatility σ, define the Sharpe ratio λ, and discuss how a fund manager would decide the proportion of his funds to be invested in the risky stock.
(ii) Describe briefly, without proofs, how to pass from the dynamics of the risky stock to the Black-Scholes formula for the price of options on it.
(iii) Why does the Black-Scholes formula not involve μ?
(iv) The Black-Scholes price does depend on σ, but we do not know it: how does one estimate σ?

Q6. (i) In the Cramér-Lundberg model of insurance claims, we treat the separate claims as independent. This is often the case, but often not the case. Comment on when this independence assumption might or might not be appropriate, with examples.
(ii) Show that, if $\kappa(s)$ is the cumulant generating function (logarithm of the moment generating function $M(s)$), then $\kappa(s)$ is convex. Explain how this is relevant to existence or otherwise of the Lundberg (adjustment) coefficient r.

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