MATL480 EXAMINATION SOLUTIONS 2017-18

Q1. (i) *Sub-prime mortgages.*

One specific trigger of the US crash in 2007 was the explosive growth in sub-prime mortgages. These were granted to people who would not have qualified as financially sound enough to get a mortgage previously, but who wanted to buy their own house. This new and profitable market proved irresistible to US banks – leading to a great house-price bubble, which burst (as bubbles do) in 2007. The knock-on effects hit the UK in 2008 (Northern Rock, etc.). The real damage of this failure of the financial sector has been its devastating and ongoing consequences on the real economy. **[8]**

(ii) *Asset-price bubbles.*

The history of stock markets (indeed, of modern capitalism in general) has typically been of periods – often quite long ones – when overall, stock prices have increased (despite the ups and downs of the business cycle, etc.), punctuated by sudden crashes, when 'the bubble bursts'. While the bubble is expanding, financier and market participants are making money (which is what they are in the market to do). No one wants to stop 'just in case'. The result is that prices have a tendency to rise way above their real value, judged by the *economic* fundamentals, rather than the *financial* market sentiment. Example: the Wall Street Crash of 1929, followed by the Slump in the US – partially ended by the Roosevelt New Deal, but only really ended by the entry of the USA into WWII after Pearl Harbour in 1941.

The Greenspan years (as Chairman of the Fed – US Federal Reserve, 1987-2006) were one long asset-price bubble, which ended in tears (in the Crash of 2007-8). The Chinese economy now has another one ... . **[9]**

(iii) *Quantitative easing (QE).*

Since the Crash, governments moved to rescue a banking system at risk of collapse by QE. The idea was to 'create electronic money' for banks to lend to businesses, to free up and kick-start the real economy, so that life could get back to normal. What tended to happen instead was that banks – also under pressure to rebuild their balance sheets – did so by holding onto this new money, rather than lending it as intended. At the same time, interest rates have been at historic lows for long periods – indeed, real (as distinct from nominal) interest rates have often been *negative*. But keeping interest rates at near-zero for long periods has itself had distorting effects – such as fuelling a new asset-price bubble, and so sewing the seeds for the next Crash. [Mainly seen – lectures] **[8]**

Q2. *Hedging.*
(i) *What is hedging? Who hedges, and why?* Hedging (useful also for *pricing* options!) is protecting oneself against loss by buying the opposite of one's position. It is typically engaged in by sellers of options. One sells an option for money, to someone who is buying insurance, and one hopes to make money from it. An option seller who remains unhedged has no protection against the financial loss involved in having the option sold exercised against him (it will not be exercised if there is no loss). His position is then *naked*, and this may be too dangerous. **[5]**
(ii) *Types of hedging.* The commonest type of hedging is *delta hedging*, using $\Delta := \partial C/\partial S$. The seller buys enough stock to offset his loss if the option is exercised against him, to first order. Similarly for the other Greeks. **[4]**
(iii) *Discrete v. continuous time.* In discrete time, one can hedge in a complete market, but in an incomplete market there may be *unhedgeable risk*. The option seller *re-balances* his portfolio at each time point.

In continuous time, this re-balancing is possible in principle. Black-Scholes markets are complete; the driving noise process is Brownian motion (BM); discounted prices are martingales under the EMM, P^* or Q. The Martingale Representation Theorem applies, and shows that option prices can be represented as Brownian integrals. The *integrand* corresponds to the *hedging strategy.* A hedger will need to re-balance continuously.

In practice, this cannot be done. For, the sample paths of BM have *infinite variation* (as their quadratic variation is finite, by Lévy's theorem). Not only would re-balancing involve an infinite amount of trading (and so infinite costs, as in reality transaction costs do exist), but would also have to be done extremely roughly. Rebalancing would be like trying to ride a bicycle, following a Brownian-like fractal path – impossible in practice. **[8]**
(iv) *Complete or partial hedging?* It depends on how the market moves (are you glad you sold the option or sorry)? To trade, one needs to *take a position* – commit funds, in the presence of uncertainty. One should not do so unless one expects to make money, at the expense of one's counter-party – who engages in the opposite trade hoping or expecting to make money out of you. To trade, one should have a judgement of where the market is going, based on knowledge and experience, and be prepared to back it. If the market moves against one, hedge to *unwind one's position* – break even from then on. In any case, one needs to know *how* to do this. **[8]**
[Mainly seen in lectures]

Q3. *Two-period binary model.*
(i) *Martingale probability.*

We determine the risk-neutral probability p^* so as to make the option a fair game [martingale]: with S_0 the initial price,

$$S_0 = p^* S_0.5/4+(1-p^*)S_0.4/5 : \ 1 = \frac{4}{5}+p^*(\frac{5}{4}-\frac{4}{5}) : \ \frac{1}{5} = p^*.\frac{9}{20} : \ p^* = \frac{4}{9}. \quad [\mathbf{5}]$$

(ii) *Pricing.* The time-2 stock prices S_2 are $S_0(5/4)^2$ (uu), S_0 (ud), $S_0.(4/5)^2$ (dd); payoffs (values) $V_2 = [S_2 - 8]_+$, which with $S_0 = 8$ are $9/2$ (uu), 0 (ud, dd). **[2]**

Work down the tree (as usual). The value V_1 at the two time-1 nodes are:

$$u-\text{node}: \qquad p^*.\frac{9}{2}+(1-p^*).0 = \frac{4}{9}.\frac{9}{2} = 2; \qquad d-\text{node}: \qquad 0. \qquad [\mathbf{4}]$$

The value of the option at time 0 is

$$V_0 = p^*.V_1(u)+(1-p^*).V_1(d) = \frac{4}{9}.2 = \frac{8}{9}. \qquad [\mathbf{4}]$$

(iii) *Hedging.*

Work up the tree (as given). From each node, the option is equivalent to ϕ_0 cash and ϕ_1 stock; the hedging portfolio is $H = (\phi_0, \phi_1)$.
Time 0.

$$u: \qquad \phi_0+\phi_1.8.\frac{5}{4} = 2, \qquad d: \qquad \phi_0+\phi_1.8.\frac{4}{5} = 0.$$

Subtract:

$$\phi_1.8.(\frac{5}{4}-\frac{4}{5}) = 2; \quad \phi_1.4.\frac{9}{20} = 1; \quad \phi_1 = \frac{5}{9};$$

$$\phi_0 = -\phi_1.8.\frac{4}{5} = -\frac{5}{9}.\frac{32}{5} = -\frac{32}{9} : \ H = (-\frac{32}{9}, \frac{5}{9}) : \text{ short } 32/9 \text{ cash, long } 5/9 \text{ stock.}$$

[5]

Time 1, d node: option worthless; $H = (0, 0)$.
Time 1, u node: stock up to 10, so (for the second time-period)

$$u: \qquad \phi_0+\phi_1.10.\frac{5}{4} = \frac{9}{2}, \qquad d: \qquad \phi_0+\phi_1.10.\frac{4}{5} = 0.$$

Subtract:

$$\phi_1.10.(\frac{5}{4}-\frac{4}{5}) = \frac{9}{2}; \quad \phi_1.10.\frac{9}{20} = \frac{9}{2}; \quad \phi_1 = 1;$$

$$\phi_0 = -\phi_1.8. = -8 : \quad H = (-8, 1) : \quad \text{short 8 cash, long 1 stock.} \qquad [\mathbf{5}]$$

[Similar seen, for the one-period case: Lectures and Problems]

Q4. *Vega.*
(i) *European calls.* With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$ the standard normal density and distribution functions, $\tau := T-t$ the time to expiry, the Black-Scholes call price is

$$C_t := S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2), \qquad (BS)$$

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} :$$

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_1-\sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} :$$

$$\phi(d_2) = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad Ke^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*) \quad [\mathbf{8}]$$

Differentiating (BS) partially w.r.t. σ gives

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial d_1/\partial\sigma - Ke^{-r\tau}\phi(d_2)\partial d_2/\partial\sigma.$$

So by $(*)$,

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial(d_1 - d_2)/\partial\sigma = S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial\sigma = S\phi(d_1)\sqrt{\tau} > 0.$$
[**6**]

European puts. Similarly, or by put-call parity, $v > 0$ here also. [**3**]

(ii) *Vega for American options.*

The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As σ increases, the Z-terms increase (vega is positive for European options). As the Zs increase, the Us increase (above: backward induction on n – DP, as usual for American options). Combining: as σ increases, the U-terms increase. So vega is also positive for American options. // [**8**]

[Seen – problems]

Q5. *Black-Scholes formula (BS).*
(a) The SDE for $GBM(\mu, \sigma)$ is $dS_t = S_t(\mu dt + \sigma dW_t)$ with $W = (W_t)$ BM. Its solution is $S_t = S_0 \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\}$. **[5]**
(b) If we change probability measure from P to P^* so as to pass from $GBM(\mu, \sigma)$ to $GBM(r, \sigma)$, and from time-interval $[0, t]$ to $[t, T]$, with W a P^*-Brownian motion we can write S_T explicitly as

$$S_T = S_t \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T - t)$, so $(W_T - W_t)/\sqrt{T - t} =: Z \sim N(0, 1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma Z\sqrt{T - t}\}, \quad s := S_t, \quad Z \sim N(0, 1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t, x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(T - t)^{\frac{1}{2}}x\} - K]_+ dx.$$

[8]

(c) To derive BS, evaluate the integral. First, [...] > 0 where

$$S_0 \exp\{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x\} > K, \qquad (r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}x > \log(K/S_0):$$

$$x > [\log(K/S_0) - (r - \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T} = c, \quad \text{say. So.}$$

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2 T}. \exp\{-\frac{1}{2}x^2 + \sigma\sqrt{T}x\}dx/\sqrt{2\pi} - Ke^{-rT}[1 - \Phi(c)],$$

and the last term is $Ke^{-rT}\Phi(-c) = Ke^{-rT}\Phi(d_-)$. The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x - \sigma\sqrt{T})^2\}dx/\sqrt{2\pi} = \int_{c-\sigma\sqrt{T}}^{\infty} \exp\{-\frac{1}{2}u^2\}du/\sqrt{2\pi}$$

$$= 1 - \Phi(c - \sigma\sqrt{T}) = \Phi(-c + \sigma\sqrt{T}) = \Phi(d_+),$$

as $-c + \sigma\sqrt{T} = d_+$ when $t = 0$. So the option price is given in terms of the initial price S_0, strike price K, expiry T, interest rate r and volatility σ by

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-), \quad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}. \quad //$$

[12]

[Seen – lectures]

Q6. *Renewal function of the exponential law.*

Recall the exponential law $E(\lambda)$: density and distribution function

$$f(x) = \lambda e^{-\lambda x}, \qquad F(x) = 1 - e^{-\lambda x} \quad (x > 0).$$

The Laplace transform of f (Laplace-Stieltjes transform, or LST, of F) is

$$\hat{F}(s) = \int_{[0,\infty)} e^{-\lambda s} dF(x) = \int_{[0,\infty)} \lambda e^{-\lambda s}.e^{-\lambda x} dx = \lambda/(\lambda + s).$$

The LST of the nth convolution F^{*n} of F is the nth power of this. Summing over n: for the renewal function $U(x)$ and its LST, $\hat{U}(s)$,

$$U(x) = \sum_{n=0}^{\infty} F^{*n}(x),$$

$$\hat{U}(s) = \sum_{n=0}^{\infty} (\lambda/(\lambda + s))^n = \frac{1}{(1 - \lambda/(\lambda + s))} = \frac{1}{s/(\lambda + s)} = 1 + \lambda/s.$$

Now with δ_0 the Dirac measure at 0 (mass 1 at the origin 0), its LST is 1. The LST of Lebesgue measure on $(0, \infty)$ (mass x on $(0, x)$) is, putting $u := sx$, $\int_0^\infty e^{-sx} dx = \int_)^\infty e^{-u} du/s = 1/s$. Combining, for $F = E(\lambda)$,

$$U(x) = \delta_0(x) + \lambda x \qquad (x \geq 0). \qquad [\mathbf{9}]$$

Interpretation. The first term 1 here means that there is always a renewal at time 0 (we always start with a new item). Then, because the hazard rate for $E(\lambda)$ is the constant λ, the expected number of renewals in $(0, x)$ is λx. [**4**]

(ii) Recall that the mean μ of $E(\lambda)$ is, putting $u := \lambda x$ as above,

$$\mu = \int_0^\infty x.f(x)dx = \int_0^\infty x.\lambda e^{-\lambda x} dx = \int_0^\infty u e^{-u} du/\lambda = 1/\lambda$$

(the integral is $\Gamma(2) = 1! = 1$). So the Renewal Theorem holds:

$$E(t) = E[N(t)] = 1 + t\lambda = 1 + t/\mu \sim t/\mu \quad (t \to \infty). \qquad [\mathbf{4}]$$

Blackwell's renewal theorem holds here with equality, as

$$U(x + h) - U(x) = h\lambda = h/\mu. \qquad [\mathbf{4}]$$

The Key Renewal Theorem holds, as

$$Z(t) = (z*U)(t) = \int_0^t z(u).u(t-u)du = \int_0^t z(u).\lambda \to \int_0^\infty z(u)du/\mu \quad (t \to \infty).$$

[**4**]

[Seen – problems] N. H. Bingham