MATL480 EXAMINATION SOLUTIONS 2016

Q1. *Reinsurance and limited liability.*
Limited liability.

Lloyd's of London pre-dates limited liability (which developed in the mid-19th C.). The Lloyd's participants, or *names*, had unlimited liability, and were liable for the full extent of losses, irrespective of their investment or their assets. This changed, following the Lloyd's scandal of the 1990s.

Insurance is now done (and most was before the Lloyd's scandal) by *limited liability companies*. So for these, the possibility or *ruin* is crucial. Not only would this wipe out the company, its assets and expertise, the jobs of its employees etc., but it would leave policy-holders without cover. **[5]**

Reinsurance. Because a run of large claims could bankrupt an insurance company, companies seek to lay off large risks – to reinsure – insure themselves – with larger, specialist reinsurance companies.

The question arises as to where reinsurance companies re-reinsure themselves ... This raises the modern form of Juvenal's question: Quis custodiet ipsos custodes – Who guards the guards? Reinsurers reinsure insurers, but – who reinsures the reinsurers? **[5]**

Regulation. It is in the interest of some industries to agree to cover each other's liabilities in the event of a bankruptcy – e.g., *travel firms*. If a travel firm goes bust, leaving large numbers of people stranded abroad, or unable to travel on a foreign holiday booked and paid for, this would destroy public confidence in the whole industry – *unless* other firms, by prior agreement, step in to cover. This is what happens, and works well.

As motor insurance is compulsory by law, motor insurance companies are regulated by the state, giving some protection against bankruptcy. **[5]**

Lender of last resort.

When a big concern is facing bankruptcy, the knock-on effects for the nation's economy may be so severe that it may be in the national interest to intervene. This is done by the *lender of last resort* – the *central bank* (Bank of England (BoE) in the UK, Federal Reserve (Fed) in the US, European Central Bank (ECB) in the European Union (EU), etc.), acting on behalf of the state (or e.g. EU). This raises questions as to the relationship between the central bank and the national government: how *independent* of government is the central bank, and so how free of *political* pressures? **[5]**

[Mainly seen – lectures]

Q2 (*Oil options*).

The price of Brent crude oil now is 150 $ per barrel. Next year, it will be 153 or 144, each with positive probability. The strike is $K = 150$.

Risk-neutral measure. We determine p^*, the 'up probability', so as to make the price a martingale. Neglecting interest, this gives

$$150 = p^*.153 + (1 - p^*).144 = 144 + 9p^*, \qquad 6 = 9p^*, \qquad p^* = 2/3.$$

(i) *Pricing.* There is no discounting, so the value V_0 at time 0 is the P^*-expectation E^* of the payoff H next year:

$$V_0 = E^*[H] = p^*.3 + (1 - p^*).0 = 3p^* = 3.2/3 = 2. \qquad [\mathbf{5}]$$

(ii) *Hedging.* The call C is financially equivalent to a portfolio Π consisting of a combination of cash and oil, as the binomial model is *complete* – all contingent claims (options etc.) can be *replicated*. To find *which* combination (ϕ_0, ϕ_1) of cash and oil, we solve two simultaneous linear equations:

$$\begin{aligned} \text{Up}: \quad & 3 = \phi_0 + 153\phi_1, \\ \text{Down}: \quad & 0 = \phi_0 + 144\phi_1. \end{aligned}$$

Subtract: $3 = 9\phi_1$: $\phi_1 = 1/3$. Substitute: $\phi_0 = -144\phi_1 = -144 \times 1/3 = -48$. So C is equivalent to the portfolio $\Pi = (-48, 1/3)$: *long*, 1/3 barrel Brent crude, *short*, $ 48 cash.

Check: in a year's time,

Oil up: Π is worth (1/3).153 - 48 = 51 - 48 = 3, as H is;

Oil down: Π is worth (1/3).144 - 48 = 48 - 48 = 0, as H is. **[5]**

(iii) *Relevant factors.* E.g., technical and geo-political factors: **[1]**

US shale oil development. The US has enormous reserves of shale oil, which can now be developed using the (novel and controversial) technique of *fracking* (hydraulic fracturing). This is environmentally damaging, so permission to use large-scale fracking is a political/legal decision, in the US (as here in the UK). Fracking is only economically worthwhile if the oil price is high. **[3]**

OPEC. The Organisation of Petroleum Exporting Countries (principally Arab countries, led by Saudi Arabia) is keen to maintain market share, and has discouraged US shale oil development by increasing its own production to keep prices low. **[3]**

Russia. The Russian economy has been hurt by Western sanctions on its oil and natural gas exports, following Russia's annexation of the Crimea and involvement in separatism in Eastern Ukraine. **[3]**

[(i), (ii): similar seen in problems; (iii): similar discussed in class]

Q3. *Brownian motion (BM).*
(i) Consider the triangular ('tent') function:

$$\Delta(t) = 2t \text{ on } [0, \frac{1}{2}), \qquad 2(1-t) \text{ on } [\frac{1}{2}, 1], \qquad 0 \text{ else.}$$

Write $\Delta_0(t) := t$, $\Delta_1(t) := \Delta(t)$ ('mother wavelet'), and define the nth *Schauder function* Δ_n ('daughter wavelets') by 'dilation and translation':

$$\Delta_n(t) := \Delta(2^j t - k) \qquad (n = 2^j + k \geq 1).$$

Then Δ_n has support $[k/2^j, (k+1)/2^j]$ (so is 'localized' on this dyadic interval – small for n, j large); (Δ_n) is a complete orthogonal system on $L^2[0,1]$. **[4]**

Theorem (PWZ theorem: Paley-Wiener-Zygmund, 1933). For $(Z_n)_0^\infty$ independent $N(0,1)$ random variables, Δ_n as above, $\lambda_n := 2^{-(j+1)/2}$,

$$W_t := \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t)$$

converges uniformly on $[0,1]$, a.s. The process $W = (W_t : t \in [0,1])$ is BM. **[4]**

(ii) **Brownian Scaling.**
For $c \in (0, \infty)$, $W(c^2 t)$ is $N(0, c^2 t)$, so $W_c(t) := c^{-1} W(c^2 t)$ is $N(0,t)$ Also

$$cov(W_c(s), W_c(t)) = c^{-2} cov(W(c^2 s), W(c^2 t)) = c^{-2} \min(c^2 s, c^2 t) = \min(s,t).$$

Thus W_c has all the defining properties of a Brownian motion: the right mean and covariance; stationary independent Gaussian increments; path-continuous; starts from 0. So, W_c **IS** a Brownian motion: if W is BM and $c > 0$, $W_c(t) := c^{-1} X(c^2 t)$, then W_c is again a BM. So W is *self-similar* (reproduces itself under scaling), so a Brownian path $W(.)$ is a *fractal.* **[6]**

(iii) *Financial modelling.* Brownian motion is the driving-noise process in the Black-Scholes model. Because of the *scaling property* above, the Black-Scholes model is insensitive to scaling. But, real markets *are* sensitive to scaling. For instance, small economic agents are price takers, while large economic agents are price makers. Also, the curvature in utility functions captures the different attitudes to a given amount of money of market participants depending on their size. This underlines one of the most important practical limitations of the Black-Scholes theory. **[6]**
[Seen in lectures]

Q4. *Geometric Brownian motion; log-prices and returns; two dimensions*
(i) *SDE.* The stochastic differential equation (SDE) for geometric Brownian motion (GBM) is

$$dS_t = S_t(\mu dt + \sigma dW_t) : \qquad dS_t/S_t = \mu dt + \sigma dW_t, \qquad (GBM)$$

with S_t the stock price, μ, σ the mean return and volatility, and (W_t) BM. **[3]**
Solution. Consider the process

$$X_t = f(t, B_t) := x_0.\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\} :$$

$$f(t,x) = x_0.\exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\}, \quad f_1 = (\mu - \frac{1}{2}\sigma^2)f, \quad f_2 = \sigma f, \quad f_{22} = \sigma^2 f.$$

By Itô's Lemma: $df = (f_1 + \frac{1}{2}f_{22})dt + f_2 dB_t$, so

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE $dX_t = X_t(\mu dt + \sigma dB_t)$, i.e. (GBM). **[4]**
(ii) *Interpretation*:

dS_t/S_t is the *return* over the time-interval $(t, t+dt)$; this is the sum of μdt, the mean return (deterministic), and σdW_t, the random component from the volatility σ and the driving noise, the BM (W_t). Thus:
returns are normally distributed. **[4]**

Thus with $Z \sim N(0,1)$ standard normal,

$$\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t : \quad \log S_t \sim \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z :$$

log-prices are normally distributed. **[3]**
(iii) *Two dimensions.*

Similarly, in two dimensions, the joint returns, or joint log-prices, are jointly normally distributed – and have the *bivariate normal* distribution, with correlation ρ, say. **[2]**

For $\rho > 0$ – two stocks in the same sector of the economy, say – one can use this to predict the conditional distribution of one given the other, as in regression. **[2]**

For $\rho < 0$: two stocks in different sectors, chosen to move against each other – balanced portfolio, as in Markowitzian diversification. **[2]**
[(i), (ii) seen, lectures; in (iii), bivariate normal and Markowitzian diversification seen]

Q5. *Poisson process; compound Poisson process.*
(i) The *Poisson process* $N = (N_t)$ of *rate* λ has stationary independent increments, and N_t is Poisson with parameter λt (so mean and variance λt). The *compound Poisson process* $CP(\lambda, F)$ is the process $S = (S_t)$, where (X_n) are independent copies with law F, independent of $N = (N_t)$, with $S_t := \sum_{n \leq N_t} X_n$. **[2, 2]**
(ii) The characteristic function (CF) of $CP(\lambda, F)$ follows from

$$\psi(u) = E[e^{iuS_t}] = E[\exp\{iu(X_1 + \ldots + X_{N_t})\}]$$
$$= \sum_n E[\exp\{iu(X_1 + \ldots + X_{N_t})\}|N_t = n].P(N_t = n)$$
$$= \sum_n e^{-\lambda t}\lambda^n t^n/n!.E[\exp\{iu(X_1 + \ldots + X_n)\}]$$
$$= \sum_n e^{-\lambda t}\lambda^n t^n/n!.(E[\exp\{iuX_1\}])^n$$
$$= \sum_n e^{-\lambda t}\lambda^n t^n/n!.\phi(u)^n = \exp\{-\lambda t(1 - \phi(u))\}. \qquad [\mathbf{5}]$$

(iii) Given N_t, $S_t = X_1 + \ldots + X_{N_t}$ has mean $N_t EX = N_t \mu$ and variance $N_t \; var \; X = N_t \sigma^2$. As N_t is Poisson with parameter λt, N_t has mean λt and variance λt. So by the Conditional Mean Formula,

$$E[S_t] = E[E[S_t|N_t]] = E[N_t \mu] = \lambda t \mu. \qquad [\mathbf{2}]$$

By the Conditional Variance Formula,

$$\begin{aligned} var\ S_t &= E[var(S_t|N_t)] + var\ E[S_t|N_t] \\ &= E[N_t var\ X] + var([N_t\ E[X]) \\ &= E[N_t].var\ X + var\ N_t.(EX)^2 \\ &= \lambda t(E[X^2] - (E[X])^2) + \lambda t.(E[X])^2 \\ &= \lambda t E[X^2] = \lambda t(\sigma^2 + \mu^2). \qquad [\mathbf{5}] \end{aligned}$$

(iv) As the convolution of two Poisson distributions $P(\lambda)$ and $P(\mu)$ is Poisson $P(\lambda + \mu)$, a Poisson distribution with large parameter is the convolution of many (Poisson) distributions, each with finite mean and variance. So by the Central Limit Theorem, it is approximately normal. So by (ii), for λt large,

$$Z := (S_t - \lambda t \mu)/\sqrt{\lambda t E[X^2]} \sim N(0, 1): \quad S_t \sim \lambda t \mu + Z\sqrt{\lambda t E[X^2]},$$

giving a normal approximation to the total-claims distribution. **[4]**
[Seen – lectures]