ullsoln3b.tex

SOLUTIONS 3b. 17.10.2018

Q1 *Doubling strategy.* (i) With N the number of losses before the first win:

$$P(N=k) = P(L, L, \cdots, L(k \text{ times}), W) = (\frac{1}{2})^k . \frac{1}{2} = (\frac{1}{2})^{k+1}.$$

That is, N is geometrically distributed with parameter 1/2. As

$$\sum_{k=0}^{\infty} P(N=k) = \sum_0^{\infty} (\frac{1}{2})^{k+1} = \frac{1}{2}/(1-\frac{1}{2}) = 1,$$

$P(N<\infty)=1$: $N<\infty$ a.s. So one is certain to win eventually.
(ii) Let S_n be one's fortune at time n. When $N=k$, one has losses at trials $1,2,3,\ldots,k$, with losses $1,2,4,\ldots,2^{k-1}$, followed by a win at trial $k+1$ (of 2^k). So one's fortune then is

$$2^k - (1+2+2^2+\ldots+2^{k-1}) = 2^k - (2^k-1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is +1 (which, by (i), one is certain to win eventually).
(iii) N has PGF

$$P(s) := E[s^N] = \sum_{n=0}^{\infty} s^k P(N=k) = \sum_0^{\infty} s^k . (\frac{1}{2})^{k+1}$$

$$= \frac{1}{2} \sum_0^{\infty} (\frac{1}{2}s)^k = \frac{1}{2}/(1-\frac{1}{2}s) = 1/(2-s):$$

$$P'(s) = E[Ns^{N-1}] = (2-s)^{-2}; \qquad P'(1) = E[N] = 1.$$

So the mean time the game lasts is 1.
(iv) As with the simple random walk (Q2 below): this is an impossible strategy to use in reality, for two reasons:
(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky.

(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice.

Q2 *First-passage time for simple random walk (SRW).*

Let $F(s) := s^T = \sum_1^\infty P(T = n)s^n = \sum_1^\infty f_n s^n$ be the PGF of T ($= T_1$, the first passage time to 1). Since the first-passage time T_2 to 2 is the sum of the first-passage times from 0 to 1 (PGF F) and from 1 to 2 (PGF F again), and these are independent (they involve disjoint blocks of independent tosses), T_2 has PGF $F_2(s) := E[s^{T_2}] = F(s)^2$.

Condition on the outcome X_1 of the first toss. If this is head $(+1)$, $T_1 = 1$. If it is a tail (-1), $T = 1 + U$, where U, the first-passage time from -1 to 1, has PGF $F_2(s) = F(s)^2$ as above. So

$$F(s) := E[s^T] = E[s^T|X_1 = +1]P(X_1 = +1) + E[s^T|X_1 = -1]P(X_1 = -1)$$
$$= \frac{1}{2}.s + \frac{1}{2}.sF(s)^2$$

(as 1 has PGF s). So F satisfies the quadratic

$$\frac{1}{2}sF(s)^2 - F(s) + \frac{1}{2}s = 0. \qquad \text{So} \qquad F(s) = \frac{1 \pm \sqrt{1-s^2}}{s}.$$

We need to take the $-$ sign here (as $F(s)$ contains no s^{-1} term):

$$F(s) = \frac{1 - \sqrt{1-s^2}}{s}.$$

(i) Put $s = 1$: $F(1) = 1$, so $\sum_1^\infty P(T = n) = 1$, so $T < \infty$ a.s.

(ii)

$$F'(s) = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} - \frac{1}{s}.\frac{\frac{1}{2}(-2s)}{\sqrt{1-s^2}} = -\frac{1}{s^2} + \frac{\sqrt{1-s^2}}{s} + \frac{1}{\sqrt{1-s^2}}.$$

So $F'(1) = E[T] = +\infty$.

(iii) In particular, $P(T = n) > 0$ for infinitely many n (indeed, for all odd n). So no bound can be put on our maximum net loss before we realise our eventual gain.

This strategy is even more unrealistic than that in Q1: it has all the disadvantages there, plus another – infinite mean waiting time.

NHB