ullsoln4a.tex

SOLUTIONS 4a. 17.10.2018

Q1. *Vega for calls.* With $\phi(x) := e^{-\frac{1}{2}x^2}/\sqrt{2\pi}$, $\Phi(x) := \int_{-\infty}^{x} \phi(u)du$ the standard normal density and distribution functions, $\tau := T - t$ the time to expiry, the Black-Scholes call price is

$$C_t := S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2), \qquad (BS)$$

$$d_1 := \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 := \frac{\log(S/K) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} :$$

$$\phi(d_2) = \phi(d_1 - \sigma\sqrt{\tau}) = \frac{e^{-\frac{1}{2}(d_1 - \sigma\sqrt{\tau})^2}}{\sqrt{2\pi}} = \frac{e^{-\frac{1}{2}d_1^2}}{\sqrt{2\pi}}.e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau} :$$

$$\phi(d_2) = \phi(d_1).e^{d_1\sigma\sqrt{\tau}}.e^{-\frac{1}{2}\sigma^2\tau}.$$

Exponentiating the definition of d_1,

$$e^{d_1\sigma\sqrt{\tau}} = (S/K).e^{r\tau}.e^{\frac{1}{2}\sigma^2\tau}.$$

Combining,

$$\phi(d_2) = \phi(d_1).(S/K).e^{r\tau} : \qquad K e^{-r\tau}\phi(d_2) = S\phi(d_1). \qquad (*)$$

Differentiating (BS) partially w.r.t. σ gives

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial d_1/\partial\sigma - K e^{-r\tau}\phi(d_2)\partial d_2/\partial\sigma.$$

So by $(*)$,

$$v := \partial C/\partial\sigma = S\phi(d_1)\partial(d_1 - d_2)/\partial\sigma = S\phi(d_1)\partial(\sigma\sqrt{\tau})/\partial\sigma = S\phi(d_1)\sqrt{\tau} > 0.$$

Vega for puts.

The same argument gives $v := \partial P/\partial\sigma > 0$, starting with the Black-Scholes formula for puts. Equivalently, we can use put-call parity

$$S + P - C = K e^{-r\tau} : \qquad \partial P/\partial\sigma = \partial C/\partial\sigma > 0.$$

Interpretation: "Options like volatility": the more uncertainty, i.e. the higher the volatility, the more the "insurance policy" of an option is worth. So vega is positive for positions *long* in the option – but negative for *short* positions.

Q2.(i) *Delta for calls.*

$$\begin{aligned}\Delta &:= \partial C/\partial S = \frac{\partial}{\partial S}[S\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)] \\ &= \Phi(d_1) + S\phi(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r\tau}\phi(d_2)\frac{\partial d_2}{\partial S} \\ &= \Phi(d_1) + S\phi(d_1)\frac{\partial(d_1 - d_2)}{\partial S},\end{aligned}$$

by Q1 $(*)$. Since $d_1 - d_2 = \sigma\sqrt{\tau}$ does not depend on S, this gives

$$\Delta = \Phi(d_1) \in (0, 1).$$

Interpretation: the payoff $(S - K)_+$ is increasing in S, so the option price should be also – and it is: $\Delta > 0$.

Also, $\Delta < 1$: options are to insure against adverse price movements. This reflects that options are useful for this: if Δ were ≥ 1, there would be no advantage in using options to hedge – we would just use a combination of cash and stock.

(ii) *Delta for puts.* Now put-call parity

$$S + P - C = Ke^{-r\tau}$$

and (i) give

$$\partial P/\partial S = \partial C/\partial S - 1 \in (-1, 0).$$

Interpretation: now the payoff $(K - S)_+$ is decreasing in S, so the option price should be also – and it is. That $\Delta > -1$ reflects that options are useful for insuring against adverse price movements (as above): if Δ were ≤ -1, we would just use a combination of cash and stock.

Q3. *Vega for American options.* The discounted value of an American option is the Snell envelope $\tilde{U}_{n-1} = \max(\tilde{Z}_{n-1}, E^*[\tilde{U}_n|\mathcal{F}_{n-1}])$ of the discounted payoff $\tilde{Z}_n$ (exercised early at time $n < N$), with terminal condition $U_N = Z_N, \tilde{U}_N = \tilde{Z}_N$. As volatility σ increases, the Zs increase: vega is positive for European options (Q1). As the Zs increase, the Us increase (above: backward induction on n – DP, as usual for American options). Combining: as σ increases, the Us increase also. So vega is also positive for American options. // NHB