

SOLUTIONS 4b. 24.10.2018

Q1.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}(x - \mu)^2/\sigma^2\right\} dx.$$

Make the substitution $u := (x - \mu)/\sigma$: $x = \mu + \sigma u$, $dx = \sigma du$:

$$M_X(t) = \int_{-\infty}^{\infty} e^{t(\mu + \sigma u)} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\} du = e^{\mu t} \cdot \int_{-\infty}^{\infty} e^{\sigma t u} \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}u^2\right\} du.$$

Completing the square in the exponent on the right,

$$\begin{aligned} M(t) &= e^{\mu t} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[u^2 - 2\sigma t u]\right\} du \\ &= e^{\mu t} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[(u - \sigma t)^2 - \sigma^2 t^2]\right\} du = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(u - \sigma t)^2\right\} du. \end{aligned}$$

The integral on the right is 1 (a density integrates to 1 – of $N(\sigma t, 1)$ as it stands, or of $N(0, 1)$ after the substitution $v := u - \sigma t$), giving

$$M(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}.$$

Q2. (i) By Q1, $M_Y(t) = E[e^{tY}] = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}$. Taking $t = 1$, $M_Y(1) = E[e^Y] = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\}$. As $X = e^Y$, this gives

$$E[X] = E[e^Y] = e^{\mu + \frac{1}{2}\sigma^2}.$$

(ii) In the Black-Scholes model, stock prices are geometric Brownian motions, driven by stochastic differential equations (with B Brownian motion)

$$dS = S(\mu dt + \sigma dB). \quad (GBM)$$

This has solution (we quote this – from Itô's lemma – Ch. V)

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}.$$

So $\log S_t = \log S_0 + (\mu - \frac{1}{2}\sigma^2)t + \sigma B_t$ is normally distributed, so S_t is lognormal.

Q3. *The exponential martingale for Brownian motion.*

The MGF of $X \sim N(\mu, \sigma^2)$ is

$$E[e^{tX}] = \exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}, \quad (*)$$

by Q1. For $B = (B_t)$ Brownian motion (BM), write

$$M_t := \exp\{\theta B_t - \frac{1}{2}\theta^2 t\}.$$

Then with $\mathcal{F} = (\mathcal{F}_t)$ the Brownian filtration, for $s \leq t$,

$$\begin{aligned} E[M_t|\mathcal{F}_s] &= E[\exp\{\theta B_t - \frac{1}{2}\theta^2 t\}|\mathcal{F}_s] \\ &= E[\exp\{\theta(B_s + (B_t - B_s)) - \frac{1}{2}\theta^2 s - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s] \\ &= \exp\{\theta B_s - \frac{1}{2}\theta^2 s\} \cdot E[\exp\{\theta(B_t - B_s) - \frac{1}{2}\theta^2(t-s)\}|\mathcal{F}_s], \end{aligned}$$

taking out what is known. The first term on the right is M_s . The conditioning in the second term can be omitted, by independent increments of BM. But $B_t - B_s \sim N(0, t-s)$, which has MGF

$$E[\exp\{\theta(B_t - B_s)\}] = \exp\{\frac{1}{2}\theta^2(t-s)\}$$

(by (*), with $\mu \mapsto 0$, $\theta^2 \mapsto t-s$, $t \mapsto \theta$). So the second term on RHS 1:

$$E[M_t|\mathcal{F}_s] = M_s.$$

So M is a martingale. //

NHB