ullsoln7b.tex

SOLUTIONS 7b. 7.11.2018

Q1. *Gamma distributions and Renewal.*
(i) *Density.*

$$\int f = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda x}.\lambda^\alpha x^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u}.u^{\alpha-1} du = 1,$$

putting $u := \lambda x$ and using the definition of the Gamma function.
(ii) *Mean.*

$$\begin{aligned}
\mu &= \int x f(x) dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x.e^{-\lambda x}.\lambda^\alpha x^{\alpha-1} dx \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty (u/\lambda).e^{-u}.u^{\alpha-1} du = \frac{1}{\lambda\Gamma(\alpha)} \int_0^\infty e^{-u}.u^\alpha du \\
&= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \\
&= \alpha/\lambda.
\end{aligned}$$

(iii) *LST*: $\hat{f}(s)$.

$$\begin{aligned}
\hat{f}(s) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-sx}.e^{-\lambda x}.\lambda^\alpha x^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda+s)x}.\lambda^\alpha x^{\alpha-1} dx \\
&= = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u} \Big(\frac{\lambda}{\lambda+s}\Big)^\alpha u^{\alpha-1} du \quad (u := (\lambda+s)x) \\
&= \Big(\frac{\lambda}{\lambda+s}\Big)^\alpha.
\end{aligned}$$

(iv) *LST*: $\hat{U}(s)$.

$$\begin{aligned}
\hat{U}(s) &= \frac{1}{1-\hat{f}(s)} = \frac{1}{1-\Big(\frac{\lambda}{\lambda+s}\Big)^\alpha} \\
&= \frac{(\lambda+s)^\alpha}{(\lambda+s)^\alpha - \lambda^\alpha} = \frac{(\lambda+s)^\alpha}{\lambda^\alpha\Big[\Big((1+\frac{s}{\lambda}\Big)^\alpha - 1\Big]}.
\end{aligned}$$

As $s \downarrow 0$,

$$(\lambda+s)^\alpha/\lambda^\alpha \to 1, \qquad \Big[\Big((1+\frac{s}{\lambda}\Big)^\alpha - 1\Big] \sim \sigma.s/\lambda = \mu s,$$

by the (generalised) Binomial Theorem (Newton: see M3H). So

$$\hat{U}(s) \sim \frac{1}{\mu s} \qquad (s \downarrow 0).$$

So by HLK with $\rho = 1$,

$$U(x) \sim x/\mu \qquad (x \to \infty),$$

giving the Renewal Theorem in this case.

NHB