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Lecture 3

Chapter 2. LÉVY PROCESSES

Characteristic Functions. To describe Lévy processes, the language of characteristic functions is essential. We review it briefly.

Definition. The *characteristic function* (CF) of a rv X is $\phi(u)$, or $\phi_X(u)$, := $E(e^{iuX})$ ($u \in \mathbf{R}$). That is, ϕ is the Fourier-Stieltjes transform $\int_{-\infty}^{\infty} e^{iux} dF(x)$ of the distribution function F of X, or the Fourier transform $\int_{-\infty}^{\infty} e^{iux} f(x) dx$ of the density f of X, when this exists.

Properties.

1. The CF *always exists.* For, $|e^{iuX}| = 1$ (u, X real). So $|E(e^{iuX})| \leq E|e^{iuX}| = E1 = 1$. So $|\phi(u)| \leq 1$: the integral/expectation defining ϕ always converges (absolutely).

2. The CF is (uniformly) continuous.

Proof.

$$\begin{aligned} |\phi(t+u) - \phi(u)| &= |E(e^{itX}.e^{iuX}) - E(e^{iuX})| \\ &= |Ee^{iuX}(e^{itX} - 1)| \\ &\leq E|e^{iuX}(e^{itX} - 1)| \\ &= E|e^{itX} - 1| \\ &\to 0 \qquad (t \to 0), \end{aligned}$$

by Lebesgue's dominated convergence theorem (which we quote from measure theory).

3. The CF determines the distribution uniquely. This is the *uniqueness theorem* for CFs (or Fourier-Stieltjes transforms), which we quote. So taking CFs loses no information.

4. *Continuity Theorem* (Lévy). If ϕ_n, ϕ are CFs, of distributions F_n, F,

(i) If $F_n \to F$ in distribution, $\phi_n(u) \to \phi(u)$ ($n \to \infty$) for all $u \in \mathbf{R}$, uniformly on compact u-sets.

(ii) If conversely $\phi_n(u)$ converges to a limit $\phi(u)$ which is *continuous at zero*, then this limit is a CF, of F say, and the corresponding distributions converge: $F_n \to F$ in distribution.

5. *Moments.* If $\mu_k := E(X^k)$ exists (i.e. if $E|X|^k < \infty$), then $\phi(u) = \sum_{j=0}^{k} (iu)^j \mu_j / j! + o(u^k)$ as $u \to 0$.

6. *Convolutions.* If X, Y are independent,

$$\phi_{X+Y}(u) = \phi_X(u).\phi_Y(u).$$

Proof. If X, Y are independent, so are e^{iuX}, e^{iuY}. So

$$\begin{aligned}\phi_{X+Y}(u) &= E(e^{iu(X+Y)}) \\ &= E(e^{iuX}.e^{iuY}) \\ &= E(e^{iuX}).E(e^{iuY})\end{aligned}$$

by the Multiplication Theorem. So the RHS is $\phi_X(u).\phi_Y(u)$.
Examples. 1. $N(0,1)$. Here

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \qquad \phi(u) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2}.e^{iux} dx = e^{-\frac{1}{2}u^2}.$$

Proof. First, replace iu (u real) by u (u real). Then the RHS is

$$\frac{1}{\sqrt{2\pi}} \int \exp\{-\frac{1}{2}(x^2 - 2ux)\} dx = \frac{1}{\sqrt{2\pi}} \int \exp\{-\frac{1}{2}(x-u)^2\}.e^{\frac{1}{2}u^2} dx.$$

Take out $e^{\frac{1}{2}u^2}$. The remaining integral is 1 (normal density integral). So

$$\frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}x^2}.e^{ux} dx = e^{\frac{1}{2}u^2}, \qquad (u \in \mathbf{R}).$$

Replacing u by iu gives the result formally. This is in fact valid by analytic continuation, which we quote from Complex Analysis.
1a. $N(\mu, \sigma^2)$.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}(x-\mu)^2/\sigma^2\}, \qquad \phi(u) = \exp\{i\mu u - \frac{1}{2}\sigma^2 u^2\}.$$

Proof. $X = \mu + \sigma Y$ with Y $N(0,1)$.
2. *Cauchy.*

$$f(x) = \frac{1}{\pi(1+x^2)}, \qquad \phi(u) = e^{-|u|}.$$

There are two ways to show this:
(a) Directly, by contour integration, Cauchy's Residue Theorem and Jordan's Lemma from Complex Analysis,
(b) Via $f(x) = \frac{1}{2}e^{-|x|} \leftrightarrow \phi(u) = 1/(1+u^2)$ (proof: integrate by parts twice), and the Fourier Integral Theorem (which again we quote).
Note. Unlike the example above, here we cannot use analytic continuation: the CF here is about as far from being analytic as it could be.

3. *Poisson*, $P(\lambda)$. $f(k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$ $(\lambda > 0)$.
$\phi(u) = \sum_{k=0}^{\infty} e^{-\lambda}\lambda^k e^{iuk}/k! = \exp\{-\lambda + \lambda e^{iu}\} = \exp\{-\lambda(1 - e^{iu})\}$.
3a. *Compound Poisson*, $CP(\lambda, F)$. Here $X_1, X_2, \ldots$ are independent and identically distributed (iid) with law F and CF ϕ, N is Poisson $P(\lambda)$ independent of (X_n), and S is the 'random sum' $S = X_1 + \ldots + X_N$. Then S has CF $E(e^{iuS}) = E(e^{iu(X_1+\ldots+X_N)})$. Conditioning on N, this is $\sum_{k=0}^{\infty} E(e^{iu(X_1+\ldots+X_N)}|N = k)P(N = k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}.E^{iu(X_1+\ldots+X_k)}$, which is $\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\phi(u)^k$ by the convolution property. The RHS is $\exp\{-\lambda + \lambda\phi(u)\} = \exp\{-\lambda(1 - \phi(u))\}$ ($P(\lambda)$ is the special case $F = \delta_1$, $\phi(u) = e^{iu}$).

Infinite Divisibility.

For $N(\mu, \sigma^2)$, the CF is

$$\phi(u) = \exp\{i\mu u - \frac{1}{2}\sigma^2 u^2\} = [\exp\{\frac{i\mu u}{n} - \frac{1}{2}\frac{\sigma^2}{n}u^2\}]^n \qquad (n = 1, 2, \ldots).$$

The RHS is the CF of the sum of n independent copies of $N(\mu/n, \sigma^2/n)$.

For Cauchy, the CF is

$$\phi(u) = e^{-|u|} = [e^{-|u|/n}]^n \qquad (n = 1, 2, \ldots).$$

The RHS is the CF of the sum of n independent random variables, each a (standard) Cauchy divided by n.

For $CP(\lambda, F)$, the CF is

$$\phi(u) = \exp\{-\lambda(1 - \phi(u))\} = [\exp\{-(\lambda/n)(1 - \phi(u))\}]^n, \qquad (n = 1, 2, \ldots),$$

the CF of the sum of n independent $CP(\lambda/n, F)$s.

In each case, we regard the distribution, or CF, as being 'divided into n pieces' (convolution factors), for each $n = 1, 2, \ldots$, in brief as being infinitely divisible:

Definition. A random variable X with distribution F and CF ϕ is *infinitely divisible* (i.d.) if for every $n = 1, 2, \ldots$, ϕ can be written as $\phi(u) = [\phi_n(u)]^n$ for some CF ϕ_n.

The Lévy-Khintchine Formula.

The form of the general infinitely-divisible distribution was studied in the 1930s by several people (including Kolmogorov and de Finetti). The final result, due to Lévy and Khintchine, is expressed in CF language - indeed, cannot be expressed otherwise.

To describe the CF of the general i.d. law, we need three components:

(i) a real a (called the *drift*, or deterministic drift),
(ii) a non-negative σ (called the diffusion coefficient, or normal component, or Gaussian component),
(iii) a (positive) measure μ on $\mathbf{R}$ (or $\mathbf{R} \setminus \{0\}$) for which

$$\int_{-\infty}^{\infty} \min(1, |x|^2)\mu(dx) < \infty,$$

that is,

$$\int_{|x|<1} |x|^2 \mu(dx) < \infty, \qquad \int_{|x|\geq 1} \mu(dx) < \infty,$$

called the Lévy measure.

THEOREM (Lévy-Khintchine Formula, L-K formula, L-K). A function ϕ is the characteristic function of an infinitely divisible distribution iff it has the form

$$\phi(u) = \exp\{-\Psi(u)\} \qquad (u \in \mathbf{R}),$$

where

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI(|x| < 1))\mu(dx) \qquad (L-K)$$

for some real a, $\sigma \geq 0$ and Lévy measure μ.

Examples.
1. $N(\mu, \sigma^2)$. Here $a = 0$, $\mu = 0$.
2. $CP(\lambda, F)$. Here $\sigma = 0$, μ has finite total mass (far from true in general!), λ say, and $\mu = \lambda F$. Then $\int_{-1}^{1} |x| d\mu(x) < \infty$, and $a = -\int_{-1}^{1} x\mu(dx)$.
3. Cauchy. See below (under 'Stable laws').

The Central Limit Problem.
Recall the classical Central Limit Theorem (CLT). If $X_1, X_2, \ldots$ are iid with mean μ and variance σ^2, $S_n = \sum_1^n X_k$, then $(S_n - n\mu)/(\sigma\sqrt{n})$ is asymptotically standard normal:

$$P\Big(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\Big) \to \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \qquad (n \to \infty) \qquad \forall x \in \mathbf{R}.$$

Exercise.
1. Prove the simpler Weak Law of Large Numbers (WLLN): If $X_1, X_2, \ldots$ are iid with mean μ, $S_n = \sum_1^n X_k$,

$$S_n/n \to \mu \qquad \text{in probability} \qquad (n \to \infty)$$

by CFs (expand the CF $\phi(u)$ in a Taylor series as far as the u term).
2. Prove the CLT by CFs (expand $\phi(u)$ as far as the u^2 term).

The CLT can be much generalized. Suppose for each $n = 1, 2, \ldots$ we have independent random variables $X_{n1}, X_{n2}, \ldots, X_{n,k_n}$, so $\{X_{nk} : 1 \leq k \leq k_n, n = 1, 2, \ldots\}$ forms a 'triangular array', and the array is 'asymtotically negligible', in that

$$\forall \epsilon > 0, \qquad P(\max_k |X_{nk}| > \epsilon) \to 0 \qquad (n \to \infty),$$

then the possible limit laws of row-sums

$$S_n := X_{n1} + \ldots + X_{n,k_n}$$

of such arrays are exactly the infinitely-divisible laws as described by (L-K). For proof, see e.g. [Gn-K], [F].

Self-Decomposability. In the central-limit problem, we used triangular - two-suffix - arrays (X_{nk}), and obtained the infinitely-divisible laws as limits. If we specialise to one-suffix arrays - *sequences* X_n of independent (not necessarily identically distributed) random variables - a subclass of the class I of infinitely-divisible laws is obtained, called the class of *self-decomposable* laws, SD. The name arises because the class SD is the class whose CFs $\phi(u)$ have the property that, for each $\rho \in (0, 1)$,

$$\phi(u) = \phi(\rho u).\phi_\rho(u),$$

where ϕ_ρ is again a CF. Thus $SD \subset I$, and the Lévy measures μ of the SD laws are those which are absolutely continuous, with density $k(x)/|x|$ where $k(x) \uparrow$ on $(-\infty, 0)$, $\downarrow$ on $(0, \infty)$ (self-decomposable laws are also called laws of Lévy's class L). For proofs and background, see e.g. [Sat], §15, §17. We shall return to the class SD in Chapter 5.

Stability. Suppose we now restrict to *identical distribution* as well as independence in SD above. That is, we seek the class of limit laws of *random walks* $S_n = \sum_1^n X_k$ with (X_n) iid - after an affine transformation (centring and scaling) - that is, for all limit laws of $(S_n - a_n)/b_n$. It turns out that the class of limit laws so obtained is the same as the class of laws for which S_n has the same *type* as X_1 - i.e. the same law to within an affine transformation, or a change of location and scale. Thus the type is 'stable' (invariant, unchanged) under addition of independent copies, whence such laws are called *stable*. They form the class S:

$$S \subset SD \subset I.$$

It turns out that this class of stable laws can be described explicitly by parameters - four in all, of which two (location and scale, specifying the law within the type) are of minor importance, leaving two essential parameters, called the *index* α ($\alpha \in (0,2]$) and the *skewness parameter* β ($\beta \in [-1,1]$). To within type, the Lévy exponent is

$$\Psi(u) = |u|^\alpha (1 - i\beta sgn(u) \tan \frac{1}{2}\pi\alpha)$$

for $\alpha \neq 1$ ($0 < \alpha < 1$ or $1 < \alpha \leq 2$) and

$$\Psi(u) = |u|(1 + i\beta sgn(u) \log |u|)$$

if $\alpha = 1$. The Lévy measure is absolutely continuous, with density of the form

$$\mu(dx) = c_+ dx/x^{1+\alpha} \qquad (x > 0), \qquad c_- dx/|x|^{1+\alpha} \qquad (x < 0),$$

with c_+, $c_- \geq 0$ and

$$\beta = (c_+ - c_-)/(c_+ + c_-).$$

For proof, see [Gn-K], [F], XVIII.6, or [Bre], §§9.8-11.

The case $\alpha = 2$ (for which β drops out) gives the normal/Gaussian case, already familiar.

The case $\alpha = 1$ and $\beta = 0$ gives the (symmetric) Cauchy law above. The case $\alpha = 1$, $\beta \neq 0$ gives the asymmetric Cauchy case, which is awkward, and we shall not pursue it.

From the form of the Lévy exponents of the remaining stable CFs (where the argument u appears only in $|u|^\alpha$ and $sgnu$), we see that, if $S_n = X_1 + \ldots + X_n$ with X_i independent copies,

$$S_n/n^{1/\alpha} = X_1 \qquad \text{in distribution} \qquad (n = 1, 2, \ldots).$$

This is called the *scaling property* of the stable laws; those (all except the asymmetric Cauchy) that possess it are called *strictly stable*.

The stable densities do not have explicit closed forms in general, only series expansions. The normal and (symmetric) Cauchy densities are known (above), as is one further important special case:

Lévy's density. Here $\alpha = 1/2$, $\beta = +1$. One can check that for each a,

$$f(x) = \frac{a}{\sqrt{2\pi x^3}} \exp\{-\frac{1}{2}a^2/x) = \frac{a}{x^{3/2}}\phi(a/\sqrt{x}) \qquad (x > 0)$$

has Laplace transform $\exp(-a\sqrt{2s})$ $(s \geq 0)$; see [R-W1] §I.9 for proof. This is the density of the first-passage time of Brownian motion (below and Ch. 3) over a level $a > 0$, as we shall see in greater generality in Ch. 4.

The other remarkable case is that of $\alpha = 3/2$, $\beta = 0$, studied by the Danish astronomer J. Holtsmark in 1919 in connection with the gravitational field of stars - this before Lévy's work on stability. The power $3/2$ comes from 3 dimensions and the inverse square law of gravity.

Exercise. Show that the Lévy measure above does indeed give the symmetric Cauchy law in the case $\alpha = 1$, $\beta = 0$. (Use symmetry to show that the relevant integral is a function of $|u|$ only, so we can take $u > 0$. Now note that the integral is real. Differentiate under the integral sign, and use $\int_0^\infty x^{-1} \sin x dx = \pi/2$.)